

UNIVERSALITY OF LOCAL STATISTICS FOR NONCOLLIDING RANDOM WALKS

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ABSTRACT. We consider the N -particle noncolliding Bernoulli random walk — a discrete time Markov process in \mathbb{Z}^N obtained from a collection of N independent simple random walks with steps $\in \{0, 1\}$ by conditioning that they never collide. We study the asymptotic behavior of local statistics of this process started from an arbitrary initial configuration on short times $T \ll N$ as $N \rightarrow +\infty$. We show that if the particle density of the initial configuration is bounded away from 0 and 1 down to scales $D \ll T$ in a neighborhood of size $Q \gg T$ of some location x (i.e., x is in the “bulk”), and the initial configuration is balanced in a certain sense, then the space-time local statistics at x are asymptotically governed by the extended discrete sine process (which can be identified with a translation invariant ergodic Gibbs measure on lozenge tilings of the plane). We also establish similar results for certain types of random initial data. Our proofs are based on a detailed analysis of the determinantal correlation kernel for the noncolliding Bernoulli random walk.

The noncolliding Bernoulli random walk is a discrete analogue of the $\beta = 2$ Dyson Brownian Motion whose local statistics are universality governed by the continuous sine process. Our results parallel the ones in the continuous case. In addition, we naturally include situations with inhomogeneous local particle density on scale T , which nontrivially affects parameters of the limiting extended sine process, and in a particular case leads to a new behavior.

1. INTRODUCTION

Our main object is a discrete time Markov chain in the N -dimensional lattice \mathbb{Z}^N which is called the *noncolliding Bernoulli random walk*. At $N = 1$ by a single-particle chain we mean the simple random walk on \mathbb{Z} which at each time step jumps by 1 in the positive direction with probability $\beta \in (0, 1)$, or stays put with the complementary probability $1 - \beta$. For $N > 1$, we consider N independent identical particles on \mathbb{Z} evolving according to the single-particle chain, and condition them to never collide (i.e., never occupy the same location of \mathbb{Z} at the same time). Note that the condition has probability zero, and therefore, needs to be defined through a limit procedure which is performed in, e.g., [KOR02] (based on a classical result of [KM59]). The result is a time and space homogeneous Markov chain $\vec{X}(t)$ living in the Weyl chamber

$$\mathbb{W}^N = \{(x_1, \dots, x_N) \in \mathbb{Z}^N : x_1 < x_2 < \dots < x_N\}, \quad (1.1)$$

with transition probabilities

$$\mathbb{P}(\vec{X}(t+1) = \vec{x}' \mid \vec{X}(t) = \vec{x}) = \begin{cases} \frac{V(\vec{x}')}{V(\vec{x})} \beta^{|\vec{x}'| - |\vec{x}|} (1 - \beta)^{N - |\vec{x}'| + |\vec{x}|}, & \text{if } x'_i - x_i \in \{0, 1\} \text{ for all } i; \\ 0, & \text{otherwise,} \end{cases} \quad (1.2)$$

where for $\vec{x} = (x_1, \dots, x_N)$ we denote $|\vec{x}| = x_1 + \dots + x_N$ and

$$V(\vec{x}) = \prod_{1 \leq j < i \leq N} (x_i - x_j). \quad (1.3)$$

We refer to Figure 1 for an illustration.

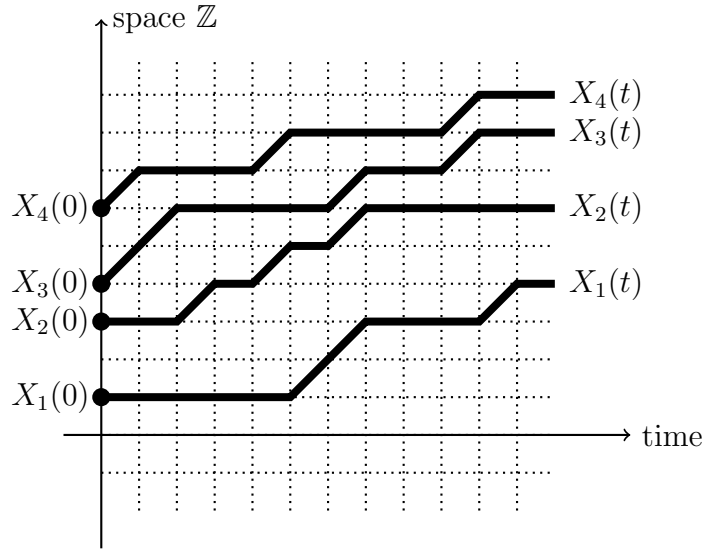


FIGURE 1. Noncolliding Bernoulli random walk of $N = 4$ particles started from the configuration $\vec{X}(0) = (1, 3, 4, 6) \in \mathbb{W}^4$.

If instead of the simple random walk we start from the Brownian Motion, then the same conditioning would lead to the celebrated ($\beta = 2$) *Dyson Brownian Motion*, which plays a prominent role in the Random Matrix Theory [Dys62], [AGZ10], [EY12].¹ Therefore, the noncolliding Bernoulli random walk can be viewed as a discrete version of the Dyson Brownian Motion. There exists also an intermediate *semi-discrete* version related to the Poisson process, see Appendix A.

On the other hand, the noncolliding Bernoulli random walk can be coupled with a $(2 + 1)$ -dimensional interacting particle system with local push/block interactions [BF14]. The latter is linked to the Totally Asymmetric Simple Exclusion Process and its relatives

¹The $\beta = 2$ Dyson Brownian Motion is naturally associated to the Gaussian Unitary Ensemble (GUE) — one of the most classical objects in the study of random matrices. There is an extension of the ensemble and of the Dyson Brownian Motion depending on a continuous parameter $\beta > 0$ (sometimes referred to as the inverse temperature). The noncolliding Brownian motions correspond to $\beta = 2$.

and to random lozenge and domino tilings. We refer to [BF14], [Nor10], [BG16], [BP14] for details.

From yet another side, fixed time distributions of $\vec{X}(t)$ can be identified with coefficients in decompositions of tensor products of certain representations of unitary groups, which are of interest in the asymptotic representation theory, see Appendix B for details.

We concentrate on the local (“bulk”) limits of $\vec{X}(T)$ as both N and T tend to infinity. More precisely, we assume that $T \ll N$, which implies that the *global* profile $\vec{X}(T)$ is almost indistinguishable from the initial condition $\vec{X}(0)$. On the other hand, our main results, Theorems 2.7 and 2.10, show that under mild conditions on $\vec{X}(0)$ (see Assumptions 1, 2 in Section 2.3), the *local* characteristics of $\vec{X}(T)$ (such as, e.g., the asymptotic distribution of the distance between two adjacent particles) become universal: they depend on two real parameters which are computed by explicit formulas involving $\vec{X}(0)$.

In more detail, we prove that the one-dimensional point process describing the particles of $\vec{X}(T)$ as $T, N \rightarrow \infty$ converges to the *discrete sine process* of [BOO00] (we recall its definition in Section 2.2). The two-dimensional point process describing the behavior of $\{\vec{X}(T+t)\}_t$ (where t is kept finite as $T \rightarrow \infty$) asymptotically becomes the *extended discrete sine process* [OR03], which can also be identified with a translation invariant ergodic Gibbs measure on lozenge tilings of the plane [She05], [KOS06].

As far as we know, in the *discrete setting* general results on the universal appearance of the discrete sine process were not available previously, and we are aware only of [Gor16] where a related theorem is proven for random lozenge tilings. However, for *specific* examples (in our context this would mean considering very special initial conditions $\vec{X}(0)$ rather than general ones; note that the existing literature was mostly dealing with *other*, yet related discrete random systems) the appearance of the extended discrete sine process was observed by many authors, cf. [BOO00], [OR03], [Joh05b], [BK08], [Bai+07], [Gor08], [BD12], [Pet14a]. We expect that our results on local behavior of the noncolliding Bernoulli random walk can serve as a step towards establishing more general bulk universality results in discrete random systems.

Comparison with Dyson Brownian Motion. In the continuous setting, the *universal* appearance of the *continuous* sine kernel process in bulk local limits of the Dyson Brownian Motion is relatively well understood. It was first conjectured by Dyson [Dys62] in the early 1960s that the universal statistics should already appear after *short* times. The first mathematical results in this direction were developed much later in [Joh01b], where the universal local behavior on *large* times was proven (using the contour integral formulas of [BH97] as an important ingredient). For the strongest results in this direction see [Shc09] and references therein.

The rigorous treatment of the short times is even more recent, with the best results appearing in [LY15], [ES15]. It should be noted that these results include cases other than the GUE ($\beta = 2$) one, and do not rely on explicit formulas specific to $\beta = 2$.

A detailed understanding of the bulk local behavior of the Dyson Brownian Motion became a crucial step towards establishing bulk universality of generalized Wigner matrices

and other random matrix ensembles, see [LY15], [ES15], [Bou+15], references therein, and the review [EY12]. See also [TV14] for an alternative approach to bulk universality of random matrices.

From this point of view, our results are parallel to the $\beta = 2$ Dyson Brownian Motion developments as we prove a discrete analogue of the Dyson's conjecture. We also provide a generalization in a different direction and study the case when the local density of particles is not restricted to be the Lebesgue measure (as was usually assumed in the study of the Dyson Brownian Motion), but can be quite general instead. This leads to new phenomena, see the end of Section 2.4 for one example.

Method. On the technical side, our approach starts from the double contour integral representation for the correlation kernel for the determinantal point process of uniformly random Gelfand–Tsetlin patterns of [Pet14a] (see also [Met13], [DM15a]). We find a limit transition which turns these random Gelfand–Tsetlin patterns into $\vec{X}(t)$, and leads to formulas for the correlation functions of the latter process. These formulas are then analyzed using the steepest descent method. For this we develop arguments working for general initial conditions $\vec{X}(0)$ rather than specific ones, and this requires a significant technical effort.

Outline. In Section 2 we formulate our main results and discuss their applications. In Section 3 we show how the noncolliding Bernoulli random walk can be obtained via a limit transition from the ensemble of uniformly random lozenge tilings of certain polygons. This leads to a double contour integral expression for the correlation kernel of the noncolliding Bernoulli random walk. Sections 4 to 6 form the main technical part of the work and are devoted to the asymptotic analysis of the correlation kernel and of the noncolliding Bernoulli random walk. In Section 7 we prove the remaining statements from Section 2 which deal with various applications of our main bulk limit theorems.

We discuss degenerations of our kernel to the kernels for noncolliding Poisson processes and for the Dyson Brownian Motion with arbitrary initial configurations in Appendix A. In Appendix B we explain a representation-theoretic interpretation of discrete-space non-colliding random walks, and formulate a more general conjecture.

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2. MAIN RESULTS

2.1. Determinantal structure. Our first result is a formula for the determinantal correlation kernel of the noncolliding Bernoulli random walk. Recall that a particle dynamics

$\vec{X}(t)$ is said to be a (dynamically) determinantal point process if its space-time correlations are given by determinants of a certain kernel $K(t, x; s, y)$:

$$\mathbb{P} \left(\text{the particle configuration } \vec{X}(t_i) \text{ on } \mathbb{Z} \text{ contains the point } y_i \text{ for all } i = 1, \dots, k \right) = \det \left[K(t_\alpha, y_\alpha; t_\beta, y_\beta) \right]_{\alpha, \beta=1}^k, \quad (2.1)$$

for any collection of pairwise distinct space-time points $(t_i, y_i) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$, $i = 1, \dots, k$. In particular, when all t_i are the same (and are equal to t), we get a determinantal point process on \mathbb{Z} with the kernel $K_t(x; y) = K(t, x; t, y)$. General details on determinantal point processes can be found in, e.g., the surveys [Sos00], [Hou+06], [Bor11].

Theorem 2.1. *The noncolliding Bernoulli random walk with parameter $\beta \in (0, 1)$ started from any initial configuration $\vec{X}(0) = \vec{a} = (a_1 < \dots < a_N) \in \mathbb{W}^N$ is determinantal in the sense of (2.1), and its correlation kernel has the following form for $x_{1,2} \in \mathbb{Z}$, $t_{1,2} \in \mathbb{Z}_{\geq 1}$:²*

$$\begin{aligned} K_{\vec{a}; \beta}^{\text{Bernoulli}}(t_1, x_1; t_2, x_2) &= \mathbf{1}_{x_1 \geq x_2} \mathbf{1}_{t_1 > t_2} (-1)^{x_1 - x_2 + 1} \binom{t_1 - t_2}{x_1 - x_2} \\ &+ \frac{t_1!}{(t_2 - 1)! (2\pi \mathbf{i})^2} \int_{x_2 - t_2 + \frac{1}{2} - i\infty}^{x_2 - t_2 + \frac{1}{2} + i\infty} dz \oint_{\text{all } w \text{ poles}} dw \frac{(z - x_2 + 1)_{t_2 - 1}}{(w - x_1)_{t_1 + 1}} \\ &\times \frac{1}{w - z} \frac{\sin(\pi w)}{\sin(\pi z)} \left(\frac{1 - \beta}{\beta} \right)^{w - z} \prod_{r=1}^N \frac{z - a_r}{w - a_r}. \end{aligned} \quad (2.2)$$

The z integration contour is the straight vertical line $\text{Re } z = x_2 - t_2 + \frac{1}{2}$ traversed upwards, and the w contour is a positively (counter-clockwise) oriented circle or a union of two circles (this depends on the ordering of t_1, x_1, t_2 , and x_2) encircling all the w poles $\{x_1 - t_1, x_1 - t_1 + 1, \dots, x_1 - 1, x_1\} \cap \{a_1, \dots, a_N\}$ of the integrand (except $w = z$), see Figure 2.

Remark 2.2. When the N -point noncolliding Bernoulli random walk starts from the densely packed configuration $\vec{a} = (0, 1, 2, \dots, N - 1)$, the distribution of the N -point configuration $\vec{X}(t) \subset \mathbb{Z}$ at any time $t \in \mathbb{Z}_{\geq 0}$ is the Krawtchouk orthogonal polynomial ensemble [KOR02]. Orthogonal polynomial ensembles are determinantal, and their correlation kernels are expressed through the corresponding univariate orthogonal polynomials — in our case, the Krawtchouk polynomials.³ This correlation kernel is explicit enough to be suitable for asymptotic analysis, see, e.g., [Joh00], [Joh01a]. The corresponding time-dependent kernel as in (2.1) is also explicitly known, it is the extended Krawtchouk kernel [Joh05b]. Theorem 2.1 generalizes these results to an arbitrary initial configuration $\vec{X}(0) = \vec{a} \in \mathbb{W}^N$.

²Throughout the text $\mathbf{1}_A$ stands for the indicator function of an event A , $\mathbf{i} = \sqrt{-1}$, and we will employ the Pochhammer symbols $(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)} = z(z+1)\dots(z+k-1)$ and the binomial coefficients $\binom{k}{a} = (-1)^a \frac{(-k)_a}{a!}$, where $k, a \in \mathbb{Z}_{\geq 0}$.

³See Section 1.10 in [KS96] for definitions and basic properties of the Krawtchouk orthogonal polynomials, and [Kön05] for a survey of orthogonal polynomial ensembles.

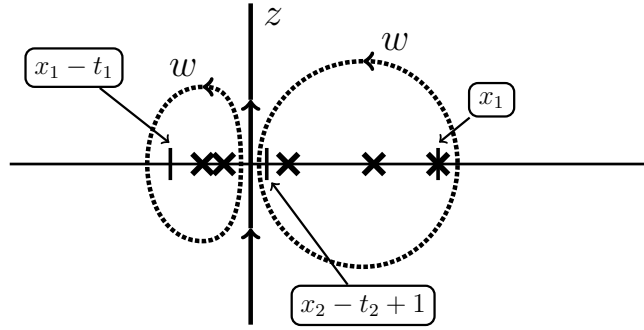


FIGURE 2. Integration contours in (2.2). The w poles are highlighted by crosses.

We prove Theorem 2.1 in Section 3 below. In Appendix A we also discuss two limits of the noncolliding Bernoulli random walk and the kernel (2.2):

- Noncolliding Poisson random walk — independent Poisson processes conditioned to never collide. This limit is obtained by rescaling time from discrete to continuous, and sending $\beta \rightarrow 0$.
- Dyson Brownian Motion — independent Brownian motions conditioned to never collide. This process (introduced in [Dys62]) is a diffusion limit of the noncolliding Bernoulli random walk. The correlation kernel for the Dyson Brownian Motion started from an arbitrary initial configuration was first obtained in [Joh01b] (see also [Shc09]). When the Dyson Brownian Motion starts from a special initial condition $(0, 0, \dots, 0)$, its determinantal correlation kernel can be expressed through the Hermite orthogonal polynomials [Meh04], [NF98].

2.2. Extended discrete sine kernel. Let us now discuss the point process describing the local asymptotic behavior the noncolliding Bernoulli random walk.

Definition 2.3. By the *extended discrete sine process*⁴ of slope $u \in \mathbb{C}$, $\text{Im}(u) > 0$, we mean the determinantal point process on $\mathbb{Z} \times \mathbb{Z}$ with the correlation kernel:

$$K_u(t, x; s, y) = -\frac{1}{2\pi i} \int_{\bar{u}}^u (1-z)^{t-s} z^{-(x-y)-1} dz, \quad (2.3)$$

where the integration path crosses $(0, 1)$ for $t > s$ and $(-\infty, 0)$ for $t \leq s$.

The extended sine process was first introduced in [OR03] in relation to the bulk limit of random lozenge tilings (equivalently, 3D Young diagrams). In that paper the kernel (2.3) was called the *incomplete beta kernel*. When $t = s$, the kernel (2.3) simplifies, after

⁴Since in this paper we only discuss convergence to the discrete sine process and do not deal with its continuous counterpart, we often drop the word “discrete”.

conjugation by the function $x \mapsto (-1)^x |u|^{-x}$,⁵ to the discrete sine kernel of [BOO00]:

$$K_\phi^{\text{sine}}(x; y) = \frac{\sin(\phi(x - y))}{\pi(x - y)}, \quad x, y \in \mathbb{Z}, \quad \phi = \pi - \arg(u), \quad (2.4)$$

with the agreement that $K_\phi^{\text{sine}}(x, x) = \phi/\pi$. The quantity $\phi/\pi \in (0, 1)$ is the density along the x direction of particles in the random configuration from the (extended) discrete sine process.

There exist other extensions of the discrete sine kernel (2.4), see [BO06], [Bor07], [BS10]. In Appendix A.1 we briefly discuss the noncolliding Poisson random walk whose local statistics should be universally governed by an extension of (2.4) other than (2.3).

Remark 2.4. The extended sine kernel was introduced in [OR03] in terms of *complementary* (to $\mathbb{Z} \times \mathbb{Z}$) configurations. The relation between kernels describing a configuration and its complement is [BOO00, Appendix A.3]

$$K^{\text{complement}}(t, x; s, y) = \mathbf{1}_{x=y} \mathbf{1}_{t=s} - K(t, x; s, y).$$

In the case of the extended sine kernel, the above delta function can be incorporated inside the contour integral by dragging for $t = s$ the contour through zero and picking the residue of $z^{-(x-y)-1}$, which is exactly $\mathbf{1}_{x=y}$.

The extended sine process is translation invariant in both directions (t and x), and it describes asymptotic bulk distribution of discrete two-dimensional determinantal point processes when both dimensions stay discrete in the limit. A characterization of the measure determined by K_z as a unique translation invariant Gibbs measure of a given complex slope was obtained in [She05], [KOS06].

2.3. Bulk limit theorems. Here we formulate our main asymptotic results — an approximation of the correlation kernel of the noncolliding Bernoulli random walk by the extended sine kernel (2.3), and a corresponding bulk local limit theorem.

Assume that N (the number of noncolliding particles) is our main parameter going to infinity, and that the time scale $T(N) \ll N$, $T(N) \rightarrow +\infty$ is fixed.⁶ For each $N = 1, 2, \dots$, we also fix an initial condition $\vec{X}(0) = \mathfrak{A}(N) = (a_1(N) < a_2(N) < \dots < a_N(N))$. We will often omit the dependence on N and simply write T (meaning $T(N)$) and a_i (meaning $a_i(N)$, $1 \leq i \leq N$), etc., when it leads to no confusion.

In what follows, we are describing the behavior of $\vec{X}(T + t)$ near the point $x = 0$. Since the definition of the non-colliding Bernoulli random walk is translation invariant, one can readily extract similar results on the behavior near an arbitrary point $x = x(N)$ by shifting $\mathfrak{A}(N)$ appropriately.

The following two assumptions will be imposed on $\mathfrak{A}(N)$ throughout the text:

⁵Transformations of a correlation kernel of the form $K(x, y) \mapsto \frac{f(x)}{f(y)} K(x, y)$ (where f is nowhere zero) not changing the correlation functions are sometimes referred to as the gauge transformations.

⁶Throughout the paper by $A(N) \ll B(N)$ we mean that $\lim_{N \rightarrow +\infty} A(N)/B(N) = 0$, and similarly for \gg .

Assumption 1 (Local density). There exist scales $D = D(N)$ satisfying $D(N) \ll T(N)$ and $Q = Q(N)$ satisfying $T(N) \ll Q(N) \ll N$, and absolute constants⁷ $0 < \rho_\bullet \leq \rho^\bullet < 1$, such that in every segment of length $D(N)$ inside $[-Q(N), Q(N)] \subset \mathbb{R}$ there are at least $\rho_\bullet D(N)$ and at most $\rho^\bullet D(N)$ points of the initial configuration $\mathfrak{A}(N)$.

Assumption 2 (Intermediate scales). For all $\delta > 0$, $R > 0$ and N large enough one has

$$\left| \sum_{i: RT(N) \leq |a_i(N)| \leq \delta N} \frac{1}{a_i(N)} \right| \leq A_{R,\delta}, \quad (2.5)$$

where $A_{R,\delta} > 0$ are absolute constants.

Remark 2.5. Note that if (2.5) holds for some $\delta_0 > 0$, $R_0 > 0$, then it holds for all other $\delta > 0$, $R > 0$ because the difference of the sums in the left-hand side of (2.5) can be bounded by a part of the harmonic series (between $R_0 T$ and RT , etc.), which is bounded by a constant independent of N .

Both Assumptions 1 and 2 serve the same goal: we want to guarantee that the average density of particles in $\vec{X}(T)$ near $x = 0$ is bounded away from 0 and from 1, as otherwise the universal local behavior might fail. Assumption 1 simply bounds the density of the initial configuration $\mathfrak{A}(N)$, while Assumption 2 requires that the “densities” of the configuration $\mathfrak{A}(N)$ far (at scales from $RT(N)$ to δN) to the right and to the left of 0 (our point of observation) are “comparable”.

We do not impose any other requirements on particles far away from $x = 0$. We remark that one can easily cook up (e.g., using an intuition coming from the study of frozen boundaries in random tiling models) a situation in which Assumption 1 fails, and yet the average density of particles in $\vec{X}(T)$ close to 0 is still bounded away from 0 and 1, and the universal local behavior holds. In other words, Assumption 1 is not necessary. On the other hand, Assumption 2 is close to being necessary, see the discussion after Theorem 2.12 below. Overall, our Assumptions 1 and 2 are simple to state and straightforward to check in applications, while a full necessary and sufficient condition would be much more technical and involved. We refer to detailed analysis in models of random tilings performed in [Pet14a], [DM15a], [DM15b], [Dus15], [Gor16].

The following function of $z \in \mathbb{C}$ will play a prominent role in our asymptotic analysis:

$$S'(z) = \sum_{r=1}^N \frac{1}{Tz - a_r} - \text{p.v.} \sum_{j \in \mathfrak{L}_T} \frac{1}{Tz - j} - \log(\beta^{-1} - 1), \quad (2.6)$$

where

$$\mathfrak{L}_T = \mathfrak{L}_{T(N)} = \{\dots, -T - 2, -T - 1, -T\} \cup \{0, 1, 2, \dots\}, \quad (2.7)$$

⁷Here and below by an absolute constant we mean a certain constant which does not depend on N , $T(N)$, or the initial configuration $\mathfrak{A}(N)$.

and the infinite sum should be understood as its principal value, i.e.,

$$\text{p.v.} \sum_{j \in \mathfrak{L}_T} \frac{1}{Tz - j} = \lim_{M \rightarrow \infty} \sum_{\substack{j \in \mathfrak{L}_T \\ |j| < M}} \frac{1}{Tz - j}. \quad (2.8)$$

We can alternatively write

$$S'(z) = \sum_{r=1}^N \frac{1}{Tz - a_r} + \sum_{i=1}^{T-1} \frac{1}{Tz + i} - \pi \cot(\pi Tz) - \log(\beta^{-1} - 1) \quad (2.9)$$

using the fact that

$$\pi \cot(\pi z) = \text{p.v.} \sum_{k \in \mathbb{Z}} \frac{1}{z - k}, \quad (2.10)$$

which follows from the Euler's product formula for the sine function

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right). \quad (2.11)$$

Proposition 2.6. *Under Assumptions 1 and 2 there exists N_0 such that for all $N > N_0$ the equation $S'(z) = 0$ has a unique root $z_c = z_c(N)$ in the upper half plane (i.e. satisfying $\text{Im}(z_c) > 0$). Moreover, there exists a compact set $\mathcal{Z} \subset \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ such that $z_c \in \mathcal{Z}$ for all $N > N_0$. The set \mathcal{Z} and the constant N_0 may depend on constants in Assumptions 1 and 2 but not on the choice of $\mathfrak{A}(N)$.*

Theorem 2.7. *Under Assumptions 1 and 2, for any fixed $t_1, x_1, t_2, x_2 \in \mathbb{Z}$ we have*

$$K_{\mathfrak{A}(N); \beta}^{\text{Bernoulli}}(t_1 + T(N), x_1; t_2 + T(N), x_2) = K_{z_c(N)/(z_c(N)+1)}(t_1, x_1; t_2, x_2) + o(1)$$

as $N \rightarrow \infty$, where $z_c(N)$ is the unique root provided by Proposition 2.6. The remainder $o(1)$ admits a tending to 0 bound which may depend on constants in Assumptions 1 and 2 but not on the choice of $\mathfrak{A}(N)$.

Remark 2.8. Since all probabilities describing the local behavior of $\vec{X}(T+t)$ near $x = 0$ (with t kept finite) are expressed through $K_{\mathfrak{A}(N); \beta}^{\text{Bernoulli}}(t_1 + T, x_1; t_2 + T, x_2)$ via (2.1), Theorem 2.7 means that as $N \rightarrow +\infty$, locally near $x = 0$ the distribution of $\{\vec{X}(T+t)\}_t$ becomes indistinguishable from the one corresponding to the extended sine process.

If $\mathfrak{A}(N)$ depends on N in a regular way, then Theorem 2.7 leads to a convergence statement.

Definition 2.9. We say that a sequence μ_k , $k = 1, 2, \dots$ of σ -finite measures on \mathbb{R} vaguely converges to μ , if for any continuous function f with compact support, we have

$$\lim_{k \rightarrow \infty} \int_{-\infty}^{+\infty} f(x) \mu_k(dx) = \int_{-\infty}^{+\infty} f(x) \mu(dx).$$

Let us also denote

$$\mathbf{d}_N(\mathbf{R}) = \sum_{i: |a_i(N)| \geq \mathbf{R}T(N)} \frac{1}{a_i(N)}. \quad (2.12)$$

Assumption 2 is equivalent to the boundedness of the $\mathbf{d}_N(\mathbf{R})$'s for fixed \mathbf{R} , uniformly in N .

Theorem 2.10. *Suppose that a sequence $\mathfrak{A}(N)$, $N = 1, 2, \dots$ is such that Assumptions 1 and 2 hold, there exists a σ -finite measure μ_{loc} for which vaguely*

$$\lim_{N \rightarrow +\infty} \frac{1}{T(N)} \sum_{i=1}^N \delta_{a_i(N)/T(N)} = \mu_{loc},$$

and there exists a limit $\mathbf{d}(\mathbf{R}) = \lim_{N \rightarrow +\infty} \mathbf{d}_N(\mathbf{R})$.

Then the point process describing $\{\vec{X}(T(N) + t)\}_t$ near $x = 0$ converges in distribution to the extended sine process of complex slope \mathbf{u}_* , in the sense that for each $t_1, x_1, t_2, x_2 \in \mathbb{Z}$ we have

$$\lim_{N \rightarrow \infty} K_{\mathfrak{A}(N); \beta}^{\text{Bernoulli}}(t_1 + T(N), x_1; t_2 + T(N), x_2) = K_{\mathbf{u}_*}(t_1, x_1; t_2, x_2),$$

where $\mathbf{u} = \mathbf{u}_* \in \mathbb{C}$ is a unique root in the upper half plane of the equation⁸

$$\int_{-\infty}^{+\infty} \left(\frac{1 - \mathbf{u}}{\mathbf{u} - (1 - \mathbf{u})v} + \frac{\mathbf{1}_{|v| > \mathbf{R}}}{v} \right) \mu_{loc}(dv) - \log \mathbf{u} = \mathbf{d}(\mathbf{R}) + \log(\beta^{-1} - 1) - \mathbf{i}\pi. \quad (2.13)$$

In fact, the additional hypotheses in Theorem 2.10 as compared to Theorem 2.7 are not too restrictive, see Remark 6.9 below.

The condition that the quantities (2.12) converge can sometimes be not easy to verify, and the determination of the limit $\mathbf{d}(\mathbf{R})$ could be even harder. Let us present a sufficient condition and a way to compute $\mathbf{d}(\mathbf{R})$ which involves the global profile $\vec{X}(0)$:

Theorem 2.11. *Suppose that a sequence $\mathfrak{A}(N)$, $N = 1, 2, \dots$ is such that Assumptions 1 and 2 hold, and, moreover, $\lim_{\delta \rightarrow 0} \mathbf{A}_{\mathbf{R}, \delta} = 0$ in Assumption 2. Next, let there exist a σ -finite measure μ_{loc} and a probability measure μ_{glob} for which vaguely*

$$\lim_{N \rightarrow +\infty} \frac{1}{T(N)} \sum_{i=1}^N \delta_{a_i(N)/T(N)} = \mu_{loc}, \quad \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N \delta_{a_i(N)/N} = \mu_{glob}, \quad (2.14)$$

and the principal value integral $\text{p.v.} \int_{-\infty}^{\infty} v^{-1} \mu_{glob}(dv)$ exists. Then the quantities $\mathbf{d}_N(\mathbf{R})$ converge to this integral (so $\mathbf{d}(\mathbf{R}) = \mathbf{d}$ is independent of \mathbf{R}), and the conclusion of Theorem 2.10 holds with

$$\mathbf{d} = \text{p.v.} \int_{-\infty}^{\infty} \frac{\mu_{glob}(dv)}{v}. \quad (2.15)$$

In many applications μ_{loc} is a multiple of the Lebesgue measure on \mathbb{R} , in which case \mathbf{u}_* is more explicit:

⁸The equation and the root \mathbf{u}_* do not depend on $\mathbf{R} > 0$.

Theorem 2.12. *Assume that Theorem 2.10 holds with μ_{loc} being $q \in (0, 1)$ times the Lebesgue measure on \mathbb{R} . Then $\mathbf{d}(\mathbf{R}) = \mathbf{d}$ does not depend on \mathbf{R} , and the complex slope of the limiting extended sine process is given by*

$$\mathbf{u}_* = \frac{\beta e^{-\mathbf{d}}}{1 - \beta} e^{i\pi(1-q)}. \quad (2.16)$$

Let us make some remarks about the elegant formula (2.16). First, the (same-time) density of particles under the limiting extended sine process is $1 - \arg(\mathbf{u}_*)/\pi = q$, as it should be. In particular, this density does not depend on the “speed” β of the noncolliding random walk, or on the parameter \mathbf{d} capturing the effect of the global profile.

To isolate the effect of the global profile, observe that the second parameter $|\mathbf{u}_*|$ of the extended sine kernel can be rewritten as

$$|\mathbf{u}_*| = \frac{\beta_{\text{eff}}}{1 - \beta_{\text{eff}}}, \quad \beta_{\text{eff}} = \frac{1}{1 + e^{\mathbf{d}}(\beta^{-1} - 1)}. \quad (2.17)$$

That is, for fixed q the bulk local distribution is the same as if the parameter β was replaced by β_{eff} , while the contribution from the global profile (encoded by \mathbf{d}) was not present.

The quantity β_{eff} increases in β and decreases in \mathbf{d} . The dependence on \mathbf{d} can be interpreted as an effect of repulsion. For example, having much more particles of the initial configuration to the right of 0 than to the left corresponds to larger values of \mathbf{d} , which leads to a decrease in β_{eff} .

Moreover, if \mathbf{d} is very large or very small, then β_{eff} is close to 0 or 1, respectively, leading to an almost deterministic behavior of the noncolliding paths in the bulk local limit. This suggests that our Assumption 2 is close to being necessary for the universal local bulk behavior. Namely, if it is violated, then $\mathbf{d} = \pm\infty$ along a subsequence $\{N_k\}$, and so the local bulk distribution is not described by the universal extended sine kernel. However, we will not pursue this analysis further.

2.4. Applications. Let us give several examples which demonstrate that Assumptions 1 and 2 are checkable in applications. The first example deals with an arbitrary *smooth* deterministic initial configuration $\vec{X}(0)$.

Theorem 2.13. *Take a twice continuously differentiable function f on $[-\frac{1}{2}, \frac{1}{2}]$ such that $f'(x) > 1$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$, and $f(-\frac{1}{2}) < 0 < f(\frac{1}{2})$. Let $\chi \in (-\frac{1}{2}, \frac{1}{2})$ be the unique point where $f(\chi) = 0$. Assume for simplicity that N is odd, and let the initial configuration of the noncolliding Bernoulli random walk be⁹*

$$a_i(N) = \lfloor Nf(i/N) \rfloor, \quad i = -\frac{N-1}{2}, -\frac{N-1}{2} + 1, \dots, \frac{N-1}{2} - 1, \frac{N-1}{2}. \quad (2.18)$$

Fix any $0 < \eta < 1$, and let $T(N) = \lfloor N^\eta \rfloor$. Then Theorem 2.12 is applicable, where μ_{loc} is the Lebesgue measure on \mathbb{R} times $q = 1/f'(\chi)$, and \mathbf{d} has the form

$$\mathbf{d} = \text{p.v.} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{f(x)}. \quad (2.19)$$

⁹Throughout the text $\lfloor \dots \rfloor$ denotes the integer part.

Remark 2.14. In the situation of Theorem 2.13 the global probability measure exists and has the form $\mu_{glob}(dv) = dv/f'(f^{-1}(v))$. The expression (2.15) for \mathbf{d} is equivalent to (2.19) via a change of variables.

The next two examples deal with *random* initial configurations $\vec{X}(0)$.

Theorem 2.15. Fix $0 < p < 1$ and $0 < \alpha < 1$. For $M = 1, 2, \dots$, consider a particle configuration on $\{-\lfloor M(1-\alpha) \rfloor, -\lfloor M(1-\alpha) \rfloor + 1, \dots, \lfloor M\alpha \rfloor - 1, \lfloor M\alpha \rfloor\}$ obtained by putting a particle at each location with probability p independently of all others. Let \mathbf{N} be the (random) number of particles in this configuration, and $\mathfrak{A}(\mathbf{N})$ denote the configuration itself. By $\vec{X}(t)$ denote the noncolliding Bernoulli random walk started from $\mathfrak{A}(\mathbf{N})$. Choose $0 < \eta < 1$ and set $T(M) = \lfloor M^\eta \rfloor$.

Then the point process $\{\vec{X}(T(M)+t)\}_t$ converges near $x = 0$ to the extended sine process as in Theorem 2.10, where μ_{loc} is p times the Lebesgue measure on \mathbb{R} , and $\mathbf{d}(\mathbf{R}) = \mathbf{d} = p \log(\frac{\alpha}{1-\alpha})$. That is, the complex slope of the limiting extended sine process is

$$u_* = \frac{\beta}{1-\beta} \left(\frac{1-\alpha}{\alpha} \right)^p e^{i\pi(1-p)}.$$

Proposition 2.16. Fix $0 < \phi < \pi$ and $0 < \alpha < 1$. For $M = 1, 2, \dots$, let the initial particle configuration of the noncolliding Bernoulli random walk be obtained by restricting the configuration of the discrete sine process of density ϕ/π (i.e., with the correlation kernel K_ϕ^{sine} given by (2.4)) to $\{-\lfloor M(1-\alpha) \rfloor, -\lfloor M(1-\alpha) \rfloor + 1, \dots, \lfloor M\alpha \rfloor - 1, \lfloor M\alpha \rfloor\}$.

With other notation the same as in Theorem 2.15, the point process $\{\vec{X}(T(M)+t)\}_t$ describing the configuration of the noncolliding walk started from the sine process initial configuration converges near $x = 0$ to the extended sine process as in Theorem 2.10, where μ_{loc} is $\frac{\phi}{\pi}$ times the Lebesgue measure on \mathbb{R} , and $\mathbf{d}(\mathbf{R}) = \mathbf{d} = \frac{\phi}{\pi} \log(\frac{\alpha}{1-\alpha})$, so

$$u_* = \frac{\beta}{1-\beta} \left(\frac{1-\alpha}{\alpha} \right)^{\frac{\phi}{\pi}} e^{i(\pi-\phi)}.$$

is the complex slope of the limiting extended sine process.

Remark 2.17. Proposition 2.16 is formulated for time $T(M) = \lfloor M^\eta \rfloor$ going to infinity. However, it is probably true even for $T(M) = 0$ because the initial configuration is already close the same-time configuration of the extended sine process. Here we do not pursue in this direction.

For the last example let us consider an initial configuration $\vec{X}(0)$ for which μ_{loc} differs from the Lebesgue measure.

Proposition 2.18. Fix two parameters $0 < \eta < 1$ and $h > 0$. Set $T(N) = \lfloor N^\eta \rfloor$. For $N = 1, 2, \dots$, let $\mathfrak{A}(N)$ be the N -particle configuration defined by the following three conditions, cf. Figure 3:

- There are $\lfloor N/2 \rfloor$ particles to the left from the origin and they occupy every second lattice site;
- $\lfloor hT(N) \rfloor$ particles adjacent to the origin from the right occupy every third lattice site;

- The remaining particles are to the right from $3\lfloor hT(N) \rfloor$ and they occupy every second lattice site.

Then the point process $\{\vec{\mathbf{X}}(T(M) + t)\}_t$ converges near $x = 0$ to the extended sine process as in Theorem 2.10. The complex slope \mathbf{u}_* of the limiting extended sine process is a unique point in the upper half-plane satisfying

$$\sqrt[6]{1 - 3h \frac{1 - \mathbf{u}_*}{\mathbf{u}_*}} = \mathbf{i} \mathbf{u}_* (1 - \beta^{-1}), \quad (2.20)$$

where the 6th degree root is understood in the sense of the principal branch.

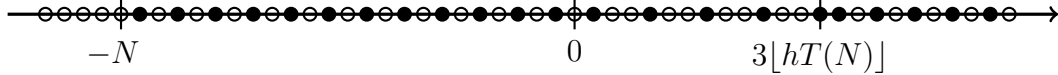


FIGURE 3. Initial condition in Proposition 2.18. Black dots mean particles.

The behavior of \mathbf{u}_* given by (2.20) as a function of h is quite interesting. When h is small, $\mathbf{u}_* \approx \mathbf{i} \frac{\beta}{1-\beta}$, matching Theorem 2.12. On the other hand, as $h \rightarrow +\infty$, \mathbf{u}_* goes to infinity in the direction $+\mathbf{i}\infty$. This leads to a behavior which seems new. Namely, as $h \rightarrow +\infty$, the same-time local configuration around zero is still governed by the discrete sine kernel (2.4) (with $\phi = \pi/2$), while the time-dependent extension of this process is *deterministic*: at each discrete time step each particle always goes to the right by 1 and does not stay put. Heuristically, for very large h the repelling force coming from the higher density region to the left of the origin is so large that this creates a deterministic flow of particles.

It is likely that with a proper time rescaling one can find a more delicate $h \rightarrow +\infty$ limit in which the paths make rare jumps, as in the classical transition from the random walk to the Poisson process. We will not pursue this direction here.

Remark 2.19. In this observation we first took the large N limit, and then a degeneration in the parameter h . This is the reason why the same-time distribution stays universal as $h \rightarrow +\infty$. We believe that if instead $h = h(N)$ goes to infinity in a certain way, then the limiting local configuration around zero would become *completely deterministic*: particles would occupy every other site, and at each time step go to the right by 1. This combined limit does not follow from Proposition 2.18, and we will not consider it further.

3. FROM LOZENGE TILINGS TO NONCOLLIDING WALKS: PROOF OF THEOREM 2.1

3.1. Random lozenge tilings. Consider uniformly random tilings (by lozenges of three types) of polygons drawn on the triangular lattice, see Figure 4, left. The asymptotic behavior of such tilings in various regimes has been studied in [CKP01], [KO07], [Ken08], [Gor08], [BGR10], [Pet14a], [Pet15], [GP15], [Nov15].¹⁰

¹⁰Note that we are using an affine transform of the regular triangular lattice, thus our picture differs from some of the references cited. This is done for a better coordinate notation in our situation.

We will employ a result of [Pet14a] (see also [DM15a]) on the determinantal structure of uniformly random lozenge tilings of polygons such as in Figure 4, left. That is, consider a trapezoid $\mathcal{T}_{N,L}$ of height $L \in \mathbb{Z}_{\geq 1}$ with vertices $(\frac{1}{2}, 0)$, $(\frac{1}{2}, L)$, $(\frac{1}{2} - N, L)$, and $(\frac{1}{2} - N - L, 0)$. Fix $(y_1 < \dots < y_L) \in \{0, -1, \dots, -N - L + 1\}$ and put lozenges of type \blacklozenge at each of the y_i 's cutting L small triangles at the base of the trapezoid. We will consider tilings of the resulting figure by lozenges of three types. The assumption that the y_i 's belong to $\{0, -1, \dots, -N - L + 1\}$ (and hence are all negative) is not essential since the whole situation is translation invariant. However, this will be convenient in Section 3.3 when discussing the limit as $L \rightarrow +\infty$.

Remark 3.1. Putting the lozenges \blacklozenge at the y_i 's fixes locations of some other of the lozenges of the same type (the darker ones in Figure 4, left). In this way the tiling we are describing can be alternatively interpreted as a lozenge tiling of a certain polygon.

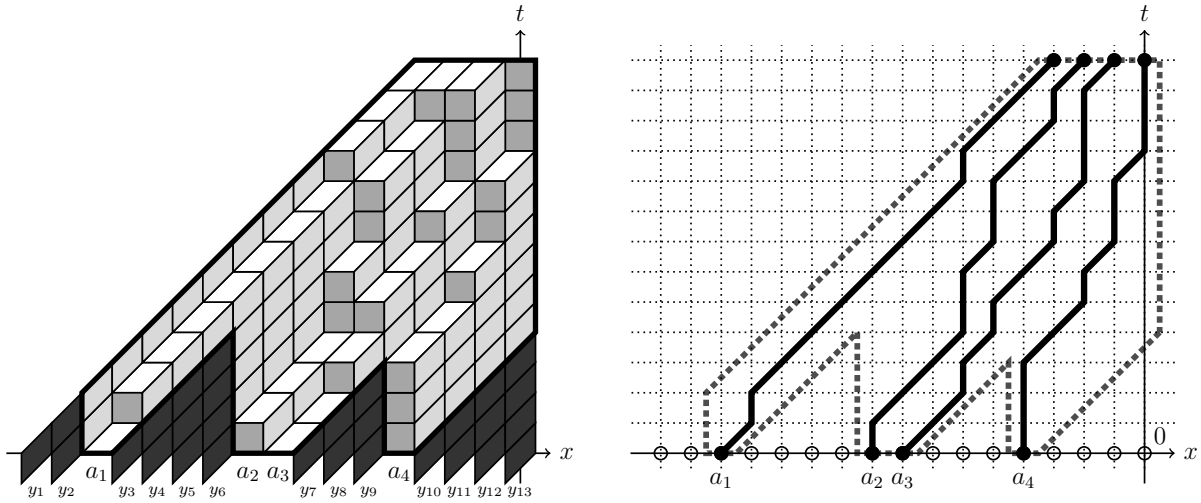


FIGURE 4. A lozenge tiling of (left) and its bijective encoding as a collection of noncolliding paths (right). The trapezoid $\mathcal{T}_{4,13}$ has height $L = 13$, and the number of paths ($N = 4$) is the same as the length of the top side of $\mathcal{T}_{4,13}$.

The total number of such tilings is equal to (e.g., see Section 2 in [BP14])

$$Z_{\{y_1, \dots, y_L\}}^{N,L} = \prod_{1 \leq i < j \leq L} \frac{y_j - y_i}{j - i}. \quad (3.1)$$

We interpret the uniformly random tiling as a random particle configuration by looking at centers of the lozenges of type \blacklozenge (there are $L(L+1)/2$ lozenges in total). The centers have integer coordinates $(x, t) \in \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0}$.

Theorem 3.2. *The uniformly random tiling described above gives rise to a determinantal point process, that is,*

$$\mathbb{P}\left(\text{there are lozenges } \blacklozenge \text{ at locations } (x_i, t_i), i = 1, \dots, k\right) \\ = \det \left[K_{N; \{y_1, \dots, y_L\}}^{\text{tilings}}(t_\alpha, x_\alpha; t_\beta, x_\beta) \right]_{\alpha, \beta=1}^k,$$

with the correlation kernel

$$K_{N; \{y_1, \dots, y_L\}}^{\text{tilings}}(t_1, x_1; t_2, x_2) = -\mathbf{1}_{t_1 < t_2} \mathbf{1}_{x_2 \leq x_1} \frac{(x_1 - x_2 + 1)_{t_2 - t_1 - 1}}{(t_2 - t_1 - 1)!} \\ + \frac{t_1!}{(t_2 - 1)! (2\pi i)^2} \oint_{c(x_2 - t_2 + 1)} dz \oint_{C(\infty)} dw \frac{(z - x_2 + 1)_{t_2 - 1}}{(w - x_1)_{t_1 + 1}} \frac{1}{w - z} \prod_{r=1}^L \frac{w - y_r}{z - y_r}. \quad (3.2)$$

Here $x_1, x_2 \in \mathbb{Z}$, $0 \leq t_1 \leq L - 1$, $1 \leq t_2 \leq L - 1$, and the integration contours look as in Figure 5:

- The z contour $c(x_2 - t_2 + 1)$ is positively oriented and encircles the points $x_2 - t_2 + 1, x_2 - t_2 + 2, \dots, y_L$, and does not encircle $x_2 - t_2, x_2 - t_2 - 1, \dots$;
- The w contour $C(\infty)$ is positively oriented and encircles $c(x_2 - t_2 + 1)$ and is sufficiently large so that to include all w poles $x_1, x_1 - 1, \dots, x_1 - t_1$.

This is the same kernel as in [Pet14a, Thm. 5.1], up to the change of coordinates $t_{1,2} = L - n_{1,2}$.

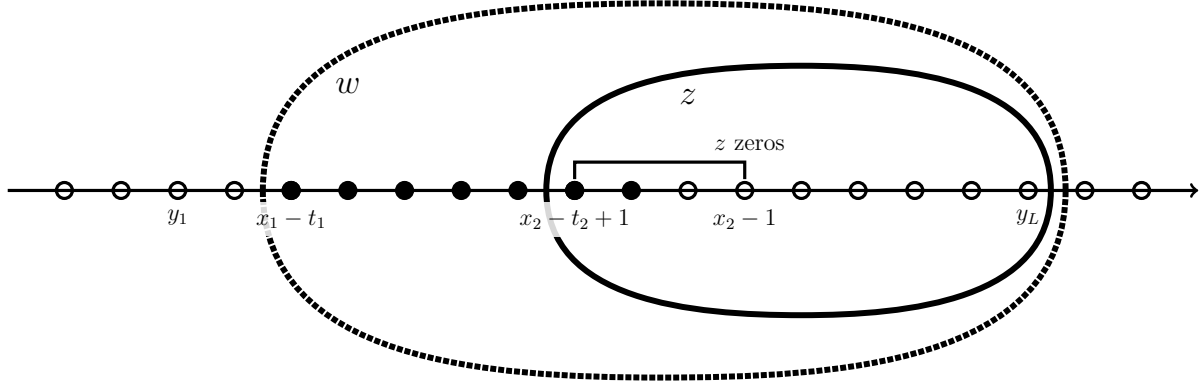


FIGURE 5. Integration contours for the kernel $K_{N; \{y_1, \dots, y_L\}}^{\text{tilings}}$ (3.2). The poles at $w = x_1 - t_1, x_1 - t_1 + 1, \dots, x_1 - 1, x_1$ are highlighted in black.

3.2. Noncolliding paths of length L . Let us keep L and y_1, \dots, y_L fixed and consider another interpretation of a lozenge tiling in terms of noncolliding paths as in Figure 4, right. There are N such paths; they trace lozenges of two types other than \blacklozenge , start at points

$$\vec{a} = (a_1 < \dots < a_N) = \{0, -1, \dots, -N - L + 1\} \setminus \{y_1, \dots, y_L\} \quad (3.3)$$

on the $t = 0$ horizontal, and end at points $-N + 1, \dots, -1, 0$, respectively, on the $t = L$ horizontal. Denote the path configuration at height t by $X_1^{(L)}(t) < \dots < X_N^{(L)}(t)$. Thus, we obtain a random path ensemble $\{\vec{X}^{(L)}(t)\}_{t=0}^L$ corresponding to our random tiling.

The correlation kernel for the paths $\vec{X}^{(L)}$ (whose configuration is complementary to the configuration of the \blacklozenge lozenges) can be obtained from the one for the tilings (3.2) by a particle-hole involution (e.g., see [BOO00, Appendix A.3]):

\mathbb{P} (paths $\vec{X}^{(L)}$ pass through points (x_i, t_i) , $i = 1, \dots, k$) = $\det [K_{L, \vec{a}}^{\text{paths}}(t_\alpha, x_\alpha; t_\beta, x_\beta)]_{\alpha, \beta=1}^k$
with

$$K_{L, \vec{a}}^{\text{paths}}(t_1, x_1; t_2, x_2) = \mathbf{1}_{x_1=x_2} \mathbf{1}_{t_1=t_2} - K_{N; \{y_1, \dots, y_L\}}^{\text{tilings}}(t_1, x_1; t_2, x_2), \quad (3.4)$$

as long as the pairwise distinct points (x_i, t_i) , $i = 1, \dots, k$, are inside the trapezoid $\mathcal{T}_{N,L}$ defined in Section 3.1 above.

Remark 3.3. The reason for the last restriction (that the observation points (x_i, t_i) belong to $\mathcal{T}_{N,L}$) is because the particle-hole involution of the configuration of lozenges gives rise not only to the N noncolliding paths $\vec{X}^{(L)}$ but also to infinitely many trivial paths connecting $(j, 0)$ to (j, L) , $j \geq 1$, and $(j, 0)$ to $(j + L, L)$, $j \leq -N - L$. These paths correspond to a unique way of extending the lozenge tiling of our polygon to the infinite horizontal strip of height L with L small triangles added at the bottom. Therefore, to capture the correlation structure of the nontrivial paths $\vec{X}^{(L)}$, the points (x_i, t_i) should be inside $\mathcal{T}_{N,L}$.

Remark 3.4. When the starting configuration \vec{a} is the densely packed one $(0, 1, \dots, N-1)$, the polygon which is tiled reduces to the hexagon. In this case the distribution of the Markov process $\{\vec{X}^{(L)}(t)\}_{t=0}^L$ is described by the extended Hahn kernel expressed through the Hahn orthogonal polynomials, see [Joh05a]. Asymptotic analysis of the noncolliding paths in the hexagon utilizing this representation of the kernel was performed in [Bai+07], [Gor08].

3.3. Limit $L \rightarrow \infty$ of noncolliding paths. We will now perform a limit transition of our uniformly random tilings to the noncolliding Bernoulli random walks. Fix $\beta \in (0, 1)$, $N \in \mathbb{Z}_{\geq 1}$, and $\vec{a} = (a_1 < \dots < a_N) \in \mathbb{W}^N$. Start the path ensemble $\{\vec{X}^{(L)}(t)\}_{t=0}^L$ from the points

$$\vec{a}^{(L)} = (a_1 - \lfloor \beta L \rfloor, \dots, a_N - \lfloor \beta L \rfloor). \quad (3.5)$$

Clearly, for L large enough we have

$$-N - L + 1 < a_i - \lfloor \beta L \rfloor < 0$$

for all i , cf. (3.3). Thus, for any fixed $\vec{a} \in \mathbb{W}^N$ the uniformly random tiling and the path ensemble $\{\vec{X}^{(L)}(t)\}_{t=0}^L$ are well-defined for large L .

The above shifting of the initial configuration of $\vec{X}^{(L)}$ forces the N noncolliding paths to have asymptotic speed β . This leads to the noncolliding Bernoulli random walk with parameter β :

Proposition 3.5. *As $L \rightarrow \infty$, all finite-dimensional distributions of the path ensemble*

$$\{\vec{X}^{(L)}(t) + \lfloor \beta L \rfloor\}_{t=0}^L$$

converge to those of the noncolliding Bernoulli random walk $\vec{X}(t)$ (defined in the Introduction) started from the configuration \vec{a} .

Proof. Because the random lozenge tiling used to construct the path ensemble $\vec{X}^{(L)}$ is picked uniformly, $\{\vec{X}^{(L)}(t)\}_{t=0}^L$ can be viewed as a Markov process (with time bounded by L). Indeed, the uniformity ensures that the past and the future are conditionally independent given the present configuration [Shi95, Ch. I.12].

Observe that each conditional probability

$$\mathbb{P}(\vec{X}^{(L)}(t+1) = \vec{b}' \mid \vec{X}^{(L)}(t) = \vec{b}), \quad \vec{b}, \vec{b}' \in \mathbb{W}^N \quad (3.6)$$

is simply the ratio of the number of lozenge tilings of a polygon of height $L - (t+1)$ with the bottom boundary determined by \vec{b}' similarly to (3.3), and the number of tilings of a polygon of height $L - t$ with the bottom boundary corresponding to \vec{b} . It suffices to show that the transition probabilities (3.6) converge to the transition probabilities (1.2) of the noncolliding Bernoulli random walk.

Let us fix $\vec{b}, \vec{b}' \in \mathbb{W}^N$ such that $b'_i - b_i \in \{0, 1\}$ for all i . (Clearly, if these conditions do not hold, then the transition probabilities from \vec{b} to \vec{b}' in both $\vec{X}^{(L)}(t) + \lfloor \beta L \rfloor$ and $\vec{X}(t)$ vanish.) We have

$$\begin{aligned} \mathbb{P}(\vec{X}^{(L)}(t+1) = \vec{b}' - \lfloor \beta L \rfloor \mid \vec{X}^{(L)}(t) = \vec{b} - \lfloor \beta L \rfloor) &= \frac{Z_{\{m'_1, \dots, m'_{L-t-1}\}}^{N, L-t-1}}{Z_{\{m_1, \dots, m_{L-t}\}}^{N, L-t}} \\ &= \prod_{1 \leq i < j \leq L-t-1} \frac{m'_j - m'_i}{j - i} \prod_{1 \leq i < j \leq L-t} \frac{j - i}{m_j - m_i} = (L-t-1)! \frac{\prod_{1 \leq i < j \leq L-t-1} (m'_j - m'_i)}{\prod_{1 \leq i < j \leq L-t} (m_j - m_i)}, \end{aligned}$$

where we used (3.1), and

$$\begin{aligned} \{m_1, \dots, m_{L-t}\} &= \{0, -1, \dots, -N - L + t + 1\} \setminus \{b_1 - \lfloor \beta L \rfloor, \dots, b_N - \lfloor \beta L \rfloor\}, \\ \{m'_1, \dots, m'_{L-t-1}\} &= \{0, -1, \dots, -N - L + t + 2\} \setminus \{b'_1 - \lfloor \beta L \rfloor, \dots, b'_N - \lfloor \beta L \rfloor\}. \end{aligned}$$

Above we have assumed that L is large enough so that all polygons are well-defined. We can write (using the notation (1.3))

$$\begin{aligned} \frac{\prod_{1 \leq i < j \leq L-t-1} (m'_j - m'_i)}{\prod_{1 \leq i < j \leq L-t} (m_j - m_i)} &= \frac{\mathbf{V}(-N - L + t + 2, \dots, -1, 0) \mathbf{V}(\vec{b}')}{\mathbf{V}(-N - L + t + 1, -N - L + t + 2, \dots, -1, 0) \mathbf{V}(\vec{b})} \\ &\quad \times \prod_{\substack{1 \leq i \leq N \\ -N - L + t + 2 \leq j \leq 0 \\ b'_i - \lfloor \beta L \rfloor \neq j}} |b'_i - \lfloor \beta L \rfloor - j|^{-1} \prod_{\substack{1 \leq i \leq N \\ -N - L + t + 1 \leq j \leq 0 \\ b_i - \lfloor \beta L \rfloor \neq j}} |b_i - \lfloor \beta L \rfloor - j| \\ &= \frac{1}{(N + L - t - 1)!} \frac{\mathbf{V}(\vec{b}')}{\mathbf{V}(\vec{b})} \prod_{i=1}^N |b_i - \lfloor \beta L \rfloor + N + L - t - 1| \prod_{\substack{1 \leq i \leq N \\ b'_i = b_i + 1}} \Pi_i, \end{aligned}$$

where

$$\begin{aligned}
\Pi_i &= \prod_{\substack{-N-L+t+2 \leq j \leq 0 \\ j \neq b_i - \lfloor \beta L \rfloor}} |b_i - \lfloor \beta L \rfloor - j| \prod_{\substack{-N-L+t+2 \leq j \leq 0 \\ j \neq b'_i - \lfloor \beta L \rfloor}} |b'_i - \lfloor \beta L \rfloor - j|^{-1} \\
&= \prod_{j=-N-L+t+2}^{b_i - \lfloor \beta L \rfloor - 1} \frac{b_i - \lfloor \beta L \rfloor - j}{b_i + 1 - \lfloor \beta L \rfloor - j} \prod_{j=b_i - \lfloor \beta L \rfloor + 2}^0 \frac{j - b_i + \lfloor \beta L \rfloor}{j - b_i - 1 + \lfloor \beta L \rfloor} \\
&= \frac{\lfloor \beta L \rfloor - b_i}{L - \lfloor \beta L \rfloor + N + b_i - t - 1}.
\end{aligned}$$

In the limit as $L \rightarrow \infty$, this converges to $\beta/(1 - \beta)$. Using the well-known asymptotics for the Gamma function [Erd53, 1.18.(5)]

$$\frac{\Gamma(L + \alpha)}{\Gamma(L)} \sim L^\alpha, \quad L \rightarrow +\infty \quad (3.7)$$

(where α is fixed), we can write for the remaining factors:

$$\lim_{L \rightarrow \infty} \frac{(L - t - 1)!}{(N + L - t - 1)!} \prod_{i=1}^N |L - \lfloor \beta L \rfloor + b_i + N - t - 1| = (1 - \beta)^N.$$

This completes the proof. \square

3.4. Limit $L \rightarrow \infty$ in the kernel. Let us now take the $L \rightarrow \infty$ limit in the kernel for the process $\vec{X}^{(L)}(t)$ coming from the uniformly random tilings. The latter kernel is given by (3.2) and (3.4).

Proposition 3.6. *Fix $\beta \in (0, 1)$ and $\vec{a} \in \mathbb{W}^N$, and consider the correlation kernel of the process $\vec{X}^{(L)}(t)$ started from the shifted initial configuration $\vec{a}^{(L)}$ as in (3.5). Then for any fixed¹¹ $t_1 \in \mathbb{Z}_{\geq 0}$, $t_2 \in \mathbb{Z}_{\geq 1}$, and $x_{1,2} \in \mathbb{Z}$ we have the convergence*

$$\lim_{L \rightarrow +\infty} K_{L; \vec{a}^{(L)}}^{\text{paths}}(t_1, x_1 - \lfloor \beta L \rfloor; t_2, x_2 - \lfloor \beta L \rfloor) = K_{\vec{a}; \beta}^{\text{Bernoulli}}(t_1, x_1; t_2, x_2),$$

where $K_{\vec{a}; \beta}^{\text{Bernoulli}}$ is given by (2.2).

Proposition 3.6 together with Proposition 3.5 will imply Theorem 2.1.

Proof of Proposition 3.6. We first focus on the part of $K_{L; \vec{a}^{(L)}}^{\text{paths}}(t_1, x_1 - \lfloor \beta L \rfloor; t_2, x_2 - \lfloor \beta L \rfloor)$ containing the double contour integral. By substituting the shifted parameters $\vec{a}^{(L)}$ and $x_{1,2} - \lfloor \beta L \rfloor$ into the kernel given by (3.2), (3.4) and at the same time shifting both integration variables z, w by $\lfloor \beta L \rfloor$ turns the double contour integral into

$$I^{\text{paths}} = \frac{t_1!}{(t_2 - 1)! (2\pi i)^2} \oint_{c(x_2 - t_2 + 1)} dz \oint_{C(\infty)} dw \frac{(z - x_2 + 1)_{t_2 - 1}}{(w - x_1)_{t_1 + 1}}$$

¹¹Clearly, under our scaling the restrictions on the variables in the kernel $K_{L; \vec{a}^{(L)}}^{\text{paths}}$ imposed in Section 3.2 (cf. Remark 3.3) will eventually disappear.

$$\times \frac{1}{w-z} \frac{(w - \lfloor \beta L \rfloor)_{N+L}}{(z - \lfloor \beta L \rfloor)_{N+L}} \prod_{r=1}^N \frac{z - a_r}{w - a_r}. \quad (3.8)$$

(Note that this integral enters $K_{L;\vec{a}(L)}^{\text{paths}}$ with a negative sign which we ignore for now.) Here the z contour $c(x_2 - t_2 + 1)$ encircles the points $x_2 - t_2 + 1, x_2 - t_2 + 2, \dots, \lfloor \beta L \rfloor$ and not $x_2 - t_2 - 1, x_2 - t_2 - 2, \dots$, while the w contour $C(\infty)$ encircles $c(x_2 - t_2 + 1)$ and all the w poles of the integrand. For large enough L these w poles are contained inside the intersection $\{x_1 - t_1, x_1 - t_1 + 1, \dots, x_1 - 1, x_1\} \cap \{a_1, \dots, a_N\}$. See Figure 6.

Let us split the w integration over $C(\infty)$ into integration over two contours: one encircling all the w poles outside the z contour $c(x_2 - t_2 + 1)$ (denote it by $c'(x_2 - t_2)$), and the other one encircling just the z contour $c(x_2 - t_2 + 1)$ (denote it by $c_{\text{out}}(x_2 - t_2 + 1)$). In this second integral we will drag the w contour inside the z contour at the cost of picking the residue at $w = z$. Denote the resulting w contour by $c_{\text{in}}(x_2 - t_2 + 1)$. See Figure 7. Thus, (3.8) can be rewritten as follows:

$$\begin{aligned} I^{\text{paths}} = & \frac{t_1!}{(t_2 - 1)! (2\pi i)^2} \oint_{c(x_2 - t_2 + 1)} dz \oint_{c'(x_2 - t_2) \cup c_{\text{in}}(x_2 - t_2 + 1)} dw \frac{1}{w - z} \frac{(z - x_2 + 1)_{t_2 - 1}}{(w - x_1)_{t_1 + 1}} \\ & \times \frac{(w - \lfloor \beta L \rfloor)_{N+L}}{(z - \lfloor \beta L \rfloor)_{N+L}} \prod_{r=1}^N \frac{z - a_r}{w - a_r} + \frac{t_1!}{(t_2 - 1)! 2\pi i} \oint_{c(x_2 - t_2 + 1)} \frac{(z - x_2 + 1)_{t_2 - 1}}{(z - x_1)_{t_1 + 1}} dz. \end{aligned} \quad (3.9)$$

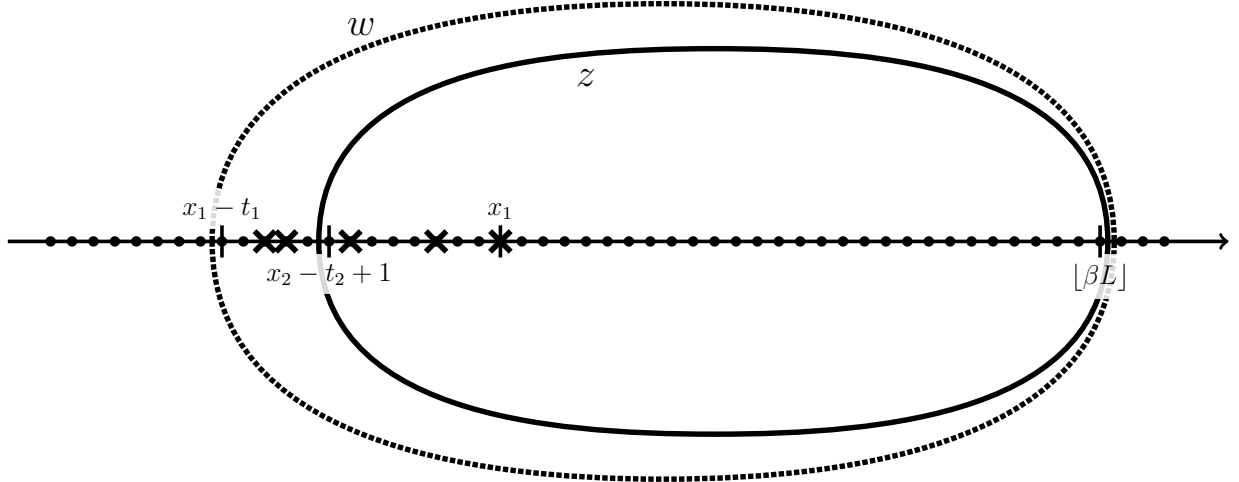


FIGURE 6. Integration contours for I^{paths} (3.8). Note that the z contour grows with L but the w poles (highlighted by crosses) do not depend on L .

The single integral in (3.9) can be evaluated using the results of Section 6.2 in [Pet14a], it is equal to

$$\mathbf{1}_{x_1 \geq x_2} \frac{(t_2 - t_1)_{x_1 - x_2}}{(x_1 - x_2)!}. \quad (3.10)$$

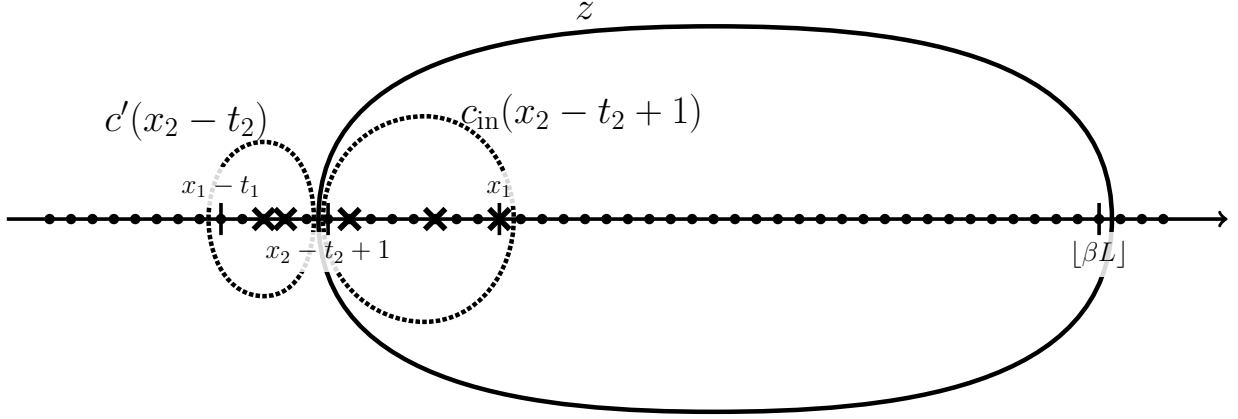


FIGURE 7. Integration contours in the double integral in (3.9).

Note that if there are no w poles inside the z contour (this happens when, e.g., $x_1 < x_2 - t_2 + 1$), then the integral over the contour $c_{\text{in}}(x_2 - t_2 + 1)$ vanishes, even though the single integral may not vanish (i.e., the dragging of the w contour is required regardless of this).

In the double contour integral in (3.9) we first note that for L fixed but large enough, due to the presence of the polynomial $(z - \lfloor \beta L \rfloor)_{N+L}$ in the denominator, the integrand decays rapidly as $z \rightarrow \infty$. Thus, the z integration contour can be replaced by the vertical line from $x_2 - t_2 + \frac{1}{2} - i\infty$ to $x_2 - t_2 + \frac{1}{2} + i\infty$ traversed from bottom to top, yielding a new minus sign in front of the double contour integral (cf. Figure 2).

The z integral over the vertical line converges uniformly in L . Indeed, observe that

$$\left| \frac{w + k}{x + iy + k} \right| < \frac{1}{1 + C|y|/|k|}, \quad k \in \mathbb{Z},$$

for some $C > 0$, where $z = x + iy$, and C is uniform in w belonging to a bounded contour. For large $|y|$ and $L > L_0$ the product of the above quantities over k as in (3.9) can be bounded by a constant independent of L times a fixed (but arbitrarily large) negative power of $|y|$. Here we also used the fact that for fixed y the infinite product over k diverges to infinity.

Thus, the integration contours do not depend on L , and we can pass to a pointwise limit as $L \rightarrow \infty$ in the integrand. Since $w, z \notin \mathbb{Z}$ on our contours, we can write

$$\begin{aligned} \frac{(w - \lfloor \beta L \rfloor)_{N+L}}{(z - \lfloor \beta L \rfloor)_{N+L}} &= \frac{\Gamma(w + L - \lfloor \beta L \rfloor + N)}{\Gamma(z + L - \lfloor \beta L \rfloor + N)} \frac{\Gamma(z - \lfloor \beta L \rfloor)}{\Gamma(w - \lfloor \beta L \rfloor)} \\ &= \frac{\Gamma(w + L - \lfloor \beta L \rfloor + N)}{\Gamma(z + L - \lfloor \beta L \rfloor + N)} \frac{\Gamma(-w + 1 + \lfloor \beta L \rfloor)}{\Gamma(-z + 1 + \lfloor \beta L \rfloor)} \frac{\sin(\pi w)}{\sin(\pi z)}, \end{aligned} \quad (3.11)$$

where in the second equality we used

$$\Gamma(u) = \frac{\pi}{\sin(\pi u) \Gamma(1 - u)}. \quad (3.12)$$

Let us employ the Stirling asymptotics for the Gamma function [Erd53, 1.18.(2)–(3)] which can be formulated as

$$\Gamma(L + \alpha) = (1 + O(L^{-1}))\sqrt{2\pi} \exp\left((L + \alpha - \tfrac{1}{2}) \log L - L\right), \quad L \rightarrow +\infty, \quad (3.13)$$

where $\alpha \in \mathbb{C}$ is fixed and the remainder $O(L^{-1})$ is uniform in α belonging to compact subsets of \mathbb{C} . Thus, continuing (3.11),

$$\frac{(w - \lfloor \beta L \rfloor)_{N+L}}{(z - \lfloor \beta L \rfloor)_{N+L}} = \frac{\sin(\pi w)}{\sin(\pi z)} \left(\frac{1 - \beta}{\beta}\right)^{w-z} (1 + O(L^{-1})).$$

Finally, the summands not involving contour integrals coming from (3.2), (3.4), and (3.10) can be simplified as

$$\begin{aligned} \mathbf{1}_{x_1=x_2} \mathbf{1}_{t_1=t_2} + \mathbf{1}_{t_1 < t_2} \mathbf{1}_{x_2 \leq x_1} \frac{(x_1 - x_2 + 1)_{t_2-t_1-1}}{(t_2 - t_1 - 1)!} - \mathbf{1}_{x_1 \geq x_2} \frac{(t_2 - t_1)_{x_1-x_2}}{(x_1 - x_2)!} \\ = \mathbf{1}_{x_1 \geq x_2} \mathbf{1}_{t_1 > t_2} (-1)^{x_1-x_2+1} \binom{t_1 - t_2}{x_1 - x_2} \end{aligned}$$

(note that all of them involve only the difference $x_1 - x_2$ which is not affected by the shift by $\lfloor \beta L \rfloor$). This coincides with the summand not containing integrals in (2.2). This completes the proof of Proposition 3.6 and hence of Theorem 2.1. \square

Remark 3.7. The argument in the above proof implies in particular that the integration in $K_{\vec{a};\beta}^{\text{Bernoulli}}$ (2.2) can be alternatively performed over a shifted contour z . This contour can be shifted as far as to the vertical line traversed from $x_2 - \frac{1}{2} - \mathbf{i}\infty$ to $x_2 - \frac{1}{2} + \mathbf{i}\infty$. Indeed, the difference between the two expressions is equal to the residue at $z = w$ integrated over a certain part of the w contour; it is the same as the single integral in (3.9) but over a contour which does not contain any poles inside, and thus vanishes.

4. SETUP OF THE ASYMPTOTIC ANALYSIS

Here we explain the relevance of the function $S'(z)$ defined in (2.6) for the asymptotics of the correlation kernel of the noncolliding Bernoulli random walks.

4.1. A change of variables. Changing the variables as $z = t_2 \mathbf{z} + x_2$, $w = t_2 \mathbf{w} + x_2$, and employing the shorthand notation

$$\Delta t = t_1 - t_2, \quad \Delta x = x_1 - x_2 \quad (4.1)$$

turns the kernel (2.2) of the noncolliding Bernoulli random walk into

$$\begin{aligned} K_{\vec{a};\beta}^{\text{Bernoulli}}(t_1, x_1; t_2, x_2) &= \mathbf{1}_{\Delta x \geq 0} \mathbf{1}_{\Delta t > 0} (-1)^{\Delta x+1} \binom{\Delta t}{\Delta x} \\ &+ \frac{1}{(2\pi \mathbf{i})^2} \int_{-1+\frac{1}{2}t_2^{-1}-\mathbf{i}\infty}^{-1+\frac{1}{2}t_2^{-1}+\mathbf{i}\infty} dz \oint_{\text{all } w \text{ poles}} dw \frac{(t_2 + \Delta t)! \cdot t_2}{(t_2 - 1)!} \frac{(t_2 \mathbf{z} + 1)_{t_2-1}}{(t_2 \mathbf{w} - \Delta x)_{t_2+\Delta t+1}} \end{aligned}$$

$$\times \frac{1}{\mathbf{w} - \mathbf{z}} \frac{\sin(\pi t_2 \mathbf{w})}{\sin(\pi t_2 \mathbf{z})} \left(\frac{1 - \beta}{\beta} \right)^{t_2(\mathbf{w} - \mathbf{z})} \prod_{r=1}^N \frac{t_2 \mathbf{z} + x_2 - a_r}{t_2 \mathbf{w} + x_2 - a_r}. \quad (4.2)$$

Here \mathbf{z} is integrated over a vertical line (which crosses the real line to the right of -1), and the \mathbf{w} integration contour (a circle or a union of two circles, cf. Figure 2) must encircle all the \mathbf{w} poles of the integrand except $\mathbf{w} = \mathbf{z}$. Note that now these poles all belong to $\{-1 + t_2^{-1}(\Delta x - \Delta t), \dots, t_2^{-1}(\Delta x - 1), t_2^{-1}\Delta x\}$.

From (4.2) we see that by shifting the initial data $\vec{a} \in \mathbb{W}^N$ it is possible to take $x_2 = 0$. Since the initial data is arbitrary and its finite shifts do not change our Assumptions 1 and 2, throughout the sequel without loss of generality we may and will assume that $x_2 = 0$, and so $x_1 = \Delta x \in \mathbb{Z}$ is fixed throughout the analysis. Moreover, since we aim to study the asymptotic behavior of $K_{\mathfrak{A}(N);\beta}^{\text{Bernoulli}}(t_1 + T(N), x_1; t_2 + T(N), 0)$ (cf. Theorem 2.7) and finite shifts in the t parameters can be incorporated into $T = T(N)$, we may also assume that $t_2 = T$ and $t_1 = \Delta t + T$, where $\Delta t \in \mathbb{Z}$ is fixed.

4.2. Definition of the function $S(\mathbf{z})$. With the notation explained in Section 4.1, rewrite the integrand in (4.2) (without $1/(\mathbf{w} - \mathbf{z})$) as follows:

$$\begin{aligned} & \frac{(T + \Delta t)! \cdot T}{(T - 1)!} \frac{(T\mathbf{z} + 1)_{T-1}}{(T\mathbf{w} - \Delta x)_{T+\Delta t+1}} \frac{\sin(\pi T\mathbf{w})}{\sin(\pi T\mathbf{z})} \left(\frac{1 - \beta}{\beta} \right)^{T(\mathbf{w} - \mathbf{z})} \prod_{r=1}^N \frac{T\mathbf{z} - a_r}{T\mathbf{w} - a_r} \\ &= \exp \left\{ T(S(\mathbf{z}) - S(\mathbf{w})) \right\} \frac{(T + \Delta t)! \cdot T}{(T - 1)!} \frac{(T\mathbf{w} + 1)_{T-1}}{(T\mathbf{w} - \Delta x)_{T+\Delta t+1}}, \end{aligned} \quad (4.3)$$

where

$$S(\mathbf{z}) = \frac{1}{T} \sum_{r=1}^N \log \left(\mathbf{z} - \frac{a_r}{T} \right) + \frac{1}{T} \sum_{i=1}^{T-1} \log \left(\mathbf{z} + \frac{i}{T} \right) - \frac{1}{T} \log(\sin(\pi T\mathbf{z})) - \mathbf{z} \log(\beta^{-1} - 1), \quad (4.4)$$

Let us discuss the choice of branches of the logarithms. Because $S(\mathbf{z})$ is exponentiated in (4.3), different choices of branches lead to the same integrand. However, a certain particular choice makes $S(\mathbf{z})$ holomorphic in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$, which will be convenient in Section 6. Let us restrict our attention to \mathbb{H} , the situation in the lower half plane is analogous (however, one clearly cannot choose a branch making $S(\mathbf{z})$ holomorphic in the whole complex plane).

The standard branch of the logarithm, denoted by $\log z$, has the cut along the negative real axis, and takes positive real values for real $z > 1$. Let $\log_{\mathbb{H}} z$ denote a branch in the upper half plane which extends holomorphically to $\mathbb{R} \setminus \{0\}$ and has the cut along the negative imaginary axis:

$$\log_{\mathbb{H}} z = \log(ze^{-i\pi/2}) + i\pi/2.$$

For $z \in \mathbb{H}$ the branches $\log z$ and $\log_{\mathbb{H}} z$ coincide. We will use $\log_{\mathbb{H}}$ for the logarithms of $\mathbf{z} - a_r/T$ and $\mathbf{z} + i/T$ in (4.4). Next, simply plugging $\sin(\pi T\mathbf{z})$ into any of these logarithms

does not produce a continuous function in \mathbb{H} . Let us use (2.11) instead, and define

$$\log(\sin \pi z)_{\mathbb{H}} = \log(\pi z) + \sum_{k=1}^{\infty} (\log(1 + z/k) + \log(1 - z/k)). \quad (4.5)$$

In the right-hand side the logarithms are standard, and we mean direct substitution. The series in k converges for any fixed $z \in \mathbb{H}$ because it is bounded by the sum of C/k^2 . One can check that alternatively (4.5) can be written as

$$\log(\sin \pi z)_{\mathbb{H}} = \log_{\mathbb{H}}(\sin \pi z) - 2\pi i \left\lfloor \frac{1}{2} \operatorname{Re}(z) + \frac{1}{2} \right\rfloor, \quad (4.6)$$

where $\log_{\mathbb{H}}(\sin \pi z)$ is the direct substitution. This expression provides a holomorphic continuation of $\log(\sin \pi z)_{\mathbb{H}}$ into $\mathbb{R} \setminus \mathbb{Z}$. From (4.6) it readily follows that

$$\log(\sin(\pi(x + iy)))_{\mathbb{H}} = -i\pi(x + iy) + i\pi/2 - \log 2 + o(1), \quad y \rightarrow +\infty, \quad (4.7)$$

uniformly in $x \in \mathbb{R}$ (the remainder $o(1)$ is periodic in x).

Therefore, the function $S(z)$ takes the form

$$\begin{aligned} S(z) = & \frac{1}{T} \sum_{r=1}^N \log_{\mathbb{H}} \left(z - \frac{a_r}{T} \right) + \frac{1}{T} \sum_{i=1}^{T-1} \log_{\mathbb{H}} \left(z + \frac{i}{T} \right) \\ & - \frac{1}{T} \log_{\mathbb{H}}(\sin(\pi Tz)) + \frac{2\pi i}{T} \left\lfloor \frac{1}{2} \operatorname{Re}(Tz) + \frac{1}{2} \right\rfloor - z \log(\beta^{-1} - 1). \end{aligned} \quad (4.8)$$

With these choices of branches it becomes holomorphic in \mathbb{H} , and extends to $z \in \mathbb{R}$ everywhere except the singularities. Recall the notation $\mathfrak{L}_T = \mathfrak{L}_{T(N)}$ (2.7) and $\mathfrak{A} = \mathfrak{A}(N)$, and denote

$$\mathfrak{l}_T = \mathfrak{l}_{T(N)} = \left\{ \dots, -1 - \frac{2}{T}, -1 - \frac{1}{T}, -1 \right\} \cup \left\{ 0, \frac{1}{T}, \frac{2}{T}, \dots \right\}, \quad \mathfrak{a} = \mathfrak{a}(N) = \left\{ \frac{a_1}{T}, \frac{a_2}{T}, \dots, \frac{a_N}{T} \right\}. \quad (4.9)$$

The set of (non-removable) singularities of $S(z)$ is $\mathfrak{l}_T \Delta \mathfrak{a}$ (the symmetric difference) because of the cancellations in (4.8) with the help of (4.5).

The function $S'(z)$ defined by (2.6) is simply the derivative of $S(z)$. Note that this derivative does not depend on choices of the branches.

We will study the asymptotic behavior of the kernel (4.2) by means of the steepest descent method. That is, we will find *critical points* of the function $S(z)$ (i.e., where $S'(z) = 0$) and deform the contours so that they pass through these critical points and are steepest descent for $\operatorname{Re} S(z)$ (that is, $\operatorname{Re} S(z)$ on these contours decreases or increases the most). As was first noted in [Oko02, Section 3.2], having a pair of nonreal complex conjugate simple critical points z_c and \bar{z}_c (plus certain properties of the integration contours) leads to the discrete sine kernel.

5. EXISTENCE OF NONREAL CRITICAL POINTS: PROOF OF PROPOSITION 2.6

In this section we deal with properties of $S'(z)$ (2.6), and prove Proposition 2.6 (stating that the equation $S'(z) = 0$ has a unique root $z_c = z_c(N)$ in \mathbb{H} , and it is uniformly bounded away from the real line and infinity) through a series of lemmas.

Lemma 5.1. *The equation $S'(z) = 0$ has at most one pair of nonreal complex conjugate roots.*

Proof. The sum over $|j| < M$ in (2.8) converges, as $M \rightarrow +\infty$, uniformly on compact sets in \mathbb{C} to the corresponding principal value sum (i.e., the left-hand side of (2.8)). Therefore, by Hurwitz's theorem, for the purpose of counting critical points it is enough to prove that the following equation (approximating $S'(z) = 0$)

$$\sum_{r=1}^N \frac{1}{Tz - a_r} - \sum_{j \in \mathfrak{L}_T \cap \{-M, \dots, M\}} \frac{1}{Tz - j} = \log(\beta^{-1} - 1), \quad (5.1)$$

has at most one pair of nonreal complex roots for all large enough M .

Let d be the size of $(\mathfrak{L}_T \Delta \mathfrak{A}) \cap \{-M, \dots, M\}$, this is the number of poles in the left-hand side of (5.1) after canceling out equal terms with opposite signs. Multiplying by the common denominator turns equation (5.1) into a polynomial equation of degree d if $\beta \neq \frac{1}{2}$, and of degree $d - 1$ otherwise (when the logarithm in the right-hand side vanishes).

Let us demonstrate that (5.1) already has at least $d - 3$ real roots. The left-hand side of (5.1) has d poles which divide the real line onto $d - 1$ segments of finite length, plus two semi-infinite rays. These d poles are of two types (see Figure 8 for an example):

- For all $a_i \in \{-T + 1, \dots, -1\}$, the pole comes from the term $\frac{1}{Tz - a_i}$.
- For points of $\{-M, \dots, -T - 1, -T\} \cup \{0, 1, \dots, M\}$ which are not equal to any a_i the pole comes from the term $-\frac{1}{Tz - \ell}$ of the opposite sign.

Clearly, on a segment between any two poles of the same sign the left-hand side of (5.1) takes all values between $-\infty$ and $+\infty$, and thus equation (5.1) has at least one root on this segment. Among the $d - 1$ segments of finite length, at most two have endpoints which are singularities of different types, and thus the presence of a root there is not guaranteed. Thus, there are at least $d - 3$ real roots. Because the coefficients of (5.1) are real, its nonreal roots come in complex conjugate pairs, and so this equation cannot have more than one such pair of nonreal roots. \square

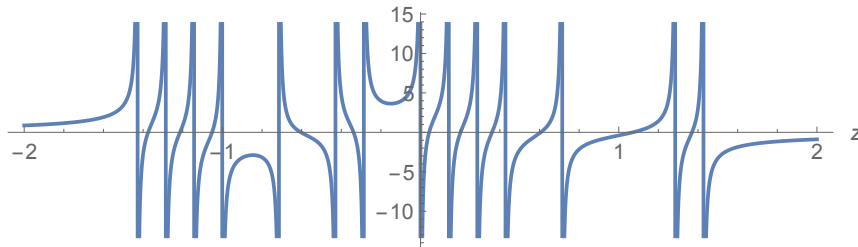


FIGURE 8. Plot of the left-hand side of (5.1) as a function of $z \in \mathbb{R}$. The parameters are $\mathfrak{A} = (-5, -3, -2, 4, 6, 7, 8)$, $T = 7$, $M = 10$, and $d = 14$. Note that for $\beta \neq \frac{1}{2}$ equation (5.1) has one extra real root belonging to one of the semi-infinite rays.

Lemma 5.1 implies that there is at most one critical point in the upper half plane. In the rest of this section we show its existence, and obtain a more precise control on the position of this critical point. The complex equation $S'(z) = 0$ (2.6) is equivalent to a pair of real equations in $z = x + iy$, $x \in \mathbb{R}$, $y \in \mathbb{R}_{>0}$:

$$0 = \operatorname{Im} S'(x + iy) = -\frac{1}{T} \sum_{r=1}^N \frac{y}{y^2 + (x - a_r/T)^2} + \frac{1}{T} \sum_{j \in \mathfrak{L}_T} \frac{y}{y^2 + (x - j/T)^2}, \quad (5.2)$$

$$0 = \operatorname{Re} S'(x + iy) = \sum_{r=1}^N \frac{1}{T} \frac{x - a_r/T}{y^2 + (x - a_r/T)^2} - \text{p.v.} \sum_{j \in \mathfrak{L}_T} \frac{1}{T} \frac{x - j/T}{y^2 + (x - j/T)^2} - \log(\beta^{-1} - 1). \quad (5.3)$$

Note that the infinite sum in (5.2) is absolutely convergent, while in (5.3) we need to use the principal value summation.

We start from (5.2), and rewrite it in a more compact form. For a discrete subset $U \subset \mathbb{R}$, define the atomic measure

$$\mathcal{M}_T[U] = \frac{1}{T} \sum_{u \in U} \delta_u \quad (5.4)$$

(note that it is not necessarily a probability or even a finite measure), and denote by

$$\mathcal{C}_y(u) = \frac{y}{\pi(y^2 + u^2)} \quad (5.5)$$

the Cauchy probability density on \mathbb{R} rescaled by $y > 0$. Using this notation, rewrite (5.2) as

$$0 = \frac{1}{\pi} \operatorname{Im} S'(x + iy) = -(\mathcal{M}_T[\mathfrak{a}] * \mathcal{C}_y)(x) + (\mathcal{M}_T[\mathfrak{l}_T] * \mathcal{C}_y)(x), \quad (5.6)$$

where “ $*$ ” means the usual convolution of measures.

Lemma 5.2. *Under Assumptions 1 and 2, for each $0 < \delta < 1$ there exists $\varepsilon_0 > 0$ (which may depend on constants in our assumptions but not on the choice of $\mathfrak{A}(N)$), such that for any $0 < \varepsilon < \varepsilon_0$ there is $N_0 \in \mathbb{Z}_{\geq 1}$, and for all $N > N_0$ (see Figure 9):*

- $\operatorname{Im} S'(x + iy) > 0$ for all (x, y) such that $\sqrt{x^2 + y^2} = \varepsilon^{-1}$ and $y \geq \varepsilon$, and for all $(x, y) = (x, \varepsilon)$, where $x \in [-\varepsilon^{-1}, -1 - \varepsilon^\delta] \cup (\varepsilon^\delta, \varepsilon^{-1}]$;
- $\operatorname{Im} S'(x + iy) < 0$ for all $(x, y) = (x, \varepsilon)$, where $x \in (-1 + \varepsilon^\delta, -\varepsilon^\delta)$.

Remark 5.3. The presence of δ in this lemma and Lemma 5.4 below is not essential. However, δ is put here to better link these statements with Lemma 5.5 below where δ plays an important role.

Proof of Lemma 5.2. Fix $\varepsilon > 0$. As N (and thus T) grows, the absolutely convergent sum $(\mathcal{M}_T[\mathfrak{l}_T] * \mathcal{C}_y)(x)$ is a Riemann sum for the corresponding integral, and it approximates the integral uniformly on compact subsets of the upper half plane (and in particular, for

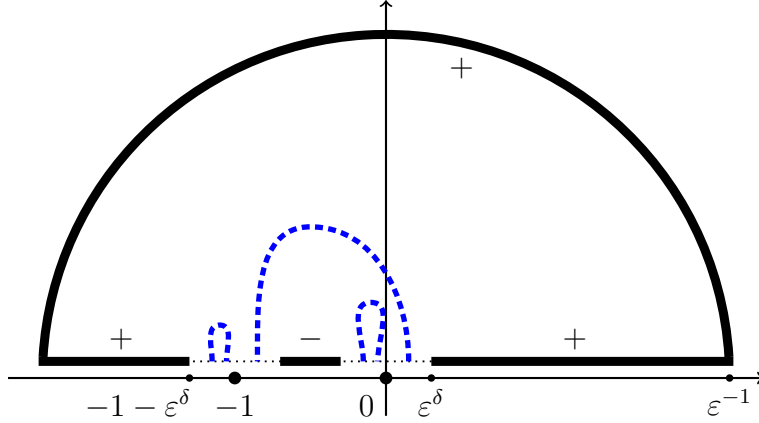


FIGURE 9. Signs of $\text{Im } S'(z)$ along the curves described in Lemma 5.2. Blue dashed curves represent a possible part of the boundary of the set \mathcal{U} inside \mathcal{D} , see Lemma 5.4 below.

(x, y) in each of the sets described in the hypotheses of the lemma). Thus, for any $c > 0$ there exists N_0 such that for all $N > N_0$,

$$\left| (\mathcal{M}_T[l_T] * \mathcal{C}_y)(x) - \frac{1}{\pi} \left(\int_{-\infty}^{-1} + \int_0^{\infty} \right) \frac{y du}{y^2 + (x - u)^2} \right| < c.$$

The integral above can be explicitly evaluated, it is equal to

$$1 + \frac{1}{\pi} \tan^{-1} \left(\frac{x}{y} \right) - \frac{1}{\pi} \tan^{-1} \left(\frac{x+1}{y} \right).$$

For small y , this expression is close to 1 if $x \in (-\infty, -1) \cup (0, +\infty)$, and close to 0 if $x \in (-1, 0)$. Moreover, for $\sqrt{x^2 + y^2} = \varepsilon^{-1}$ and $y \geq \varepsilon$ this expression is close to 1, too.

Let us now deal with $(\mathcal{M}_T[\mathbf{a}] * \mathcal{C}_y)(x)$, which enters (5.2) with a negative sign. We aim to show that this sum is bounded away from 0 and 1, which will imply the claim. Use Assumption 1 and take N so large that $\mathbf{Q} > 2T\varepsilon^{-1}$. Throw away summands for which $|a_i| > 2T\varepsilon^{-1}$, and then split the segment $(-2T\varepsilon^{-1}, 2T\varepsilon^{-1})$ into $4T\varepsilon^{-1}/D$ segments of the form $(-2T\varepsilon^{-1} + jD, -2T\varepsilon^{-1} + (j+1)D)$, each of which contains at least $\rho_{\bullet}D$ points from the configuration \mathfrak{A} . On each of these segments, replace the summands $\frac{1}{\pi T} \frac{y}{y^2 + (x - a_r/T)^2}$ by $\rho_{\bullet}D$ times the minimum of $\frac{1}{\pi T} \frac{y}{y^2 + (x - a/T)^2}$ over a belonging to the corresponding segment. This allows to estimate $(\mathcal{M}_T[\mathbf{a}] * \mathcal{C}_y)(x)$ from below by a Riemann sum of the integral

$$\rho_{\bullet} \int_{-2\varepsilon^{-1}}^{2\varepsilon^{-1}} \frac{1}{\pi} \frac{y du}{y^2 + (x - u)^2} = \frac{\rho_{\bullet}}{\pi} \left[\tan^{-1} \left(\frac{2\varepsilon^{-1} - x}{y} \right) + \tan^{-1} \left(\frac{2\varepsilon^{-1} + x}{y} \right) \right]$$

within error $O(T^{-1}\varepsilon^{-1})$ which goes to zero. For $y = \varepsilon$, the expression in the square brackets is close to π , and for $\sqrt{x^2 + y^2} = \varepsilon^{-1}$ and $y \geq \varepsilon$ it is $\geq \frac{\pi}{2}$. Therefore, $(\mathcal{M}_T[\mathbf{a}] * \mathcal{C}_y)(x) \geq \frac{\rho_{\bullet}}{2}$.

The other estimate is obtained in a similar manner but now we assume that all locations outside $(-2T\varepsilon^{-1}, 2T\varepsilon^{-1})$ are occupied by particles from the configuration \mathfrak{A} . This allows

to write

$$\begin{aligned} (\mathcal{M}_T[\mathbf{a}] * \mathcal{C}_y)(x) &\leq \frac{\rho^\bullet}{\pi} \int_{-2\varepsilon^{-1}}^{2\varepsilon^{-1}} \frac{y du}{y^2 + (x-u)^2} + \int_{\mathbb{R} \setminus (-2\varepsilon^{-1}, 2\varepsilon^{-1})} \frac{1}{\pi} \frac{y du}{y^2 + (x-u)^2} + O\left(\frac{1}{T\varepsilon}\right) \\ &= \frac{\rho^\bullet - 1}{\pi} \left[\tan^{-1} \left(\frac{2\varepsilon^{-1} - x}{y} \right) + \tan^{-1} \left(\frac{2\varepsilon^{-1} + x}{y} \right) \right] + 1 + O\left(\frac{1}{T\varepsilon}\right) \leq \frac{1 + \rho^\bullet}{2} \end{aligned}$$

for large enough N . This completes the proof. \square

Lemma 5.4. *Under Assumptions 1 and 2, for each $0 < \delta < 1$ there exists $\varepsilon_0 > 0$ (which may depend on constants in our assumptions but not on the choice of $\mathfrak{A}(N)$), such that for any $0 < \varepsilon < \varepsilon_0$ there is $N_0 \in \mathbb{Z}_{\geq 1}$, and for all $N > N_0$ there exists a curve $\gamma = \gamma(N)$ in the upper half plane with the following properties:*

- For all $z \in \gamma$ we have $\text{Im } S'(z) = 0$, $\text{Im}(z) \geq \varepsilon$, and $|z| < \varepsilon^{-1}$;
- The curve γ starts in the set $\{x + i\varepsilon : -1 - \varepsilon^\delta < x < -1 + \varepsilon^\delta\}$, and ends in the set $\{x + i\varepsilon : -\varepsilon^\delta < x < \varepsilon^\delta\}$.

Proof. Let $\mathcal{D} = \{x + iy \in \mathbb{C} : y > \varepsilon, \sqrt{x^2 + y^2} < \varepsilon^{-1}\}$, and denote

$$\mathcal{U} = \mathcal{D} \cap \{x + iy \in \mathbb{C} : \text{Im } S'(x + iy) < 0\}.$$

By Lemma 5.2, the part of the boundary of \mathcal{U} which lies inside the interior of \mathcal{D} is a union of several curves whose start and end points belong to

$$\{x + i\varepsilon \in \mathbb{C} : -1 - \varepsilon^\delta < x < -1 + \varepsilon^\delta \text{ or } -\varepsilon^\delta < x < \varepsilon^\delta\},$$

cf. Figure 9.¹²

By continuity and Lemma 5.2, on any path from the segment $\{x + i\varepsilon : -1 + \varepsilon^\delta < x < -\varepsilon^\delta\}$ (where $\text{Im } S'(z) < 0$) to the curved boundary of \mathcal{D} (where $\text{Im } S'(z) > 0$) there exists a point where $\text{Im } S'(z) = 0$. Thus, as γ we can take any of the curves forming the boundary of \mathcal{U} inside \mathcal{D} which starts to the left of $-1 + \varepsilon^\delta$, ends to the right of $-\varepsilon^\delta$, and does not intersect the set $\{x + i\varepsilon\}$ except at its endpoints. This implies the claim. \square

Lemma 5.5. *Under Assumptions 1 and 2, there exist $0 < \delta < 1$ and $\varepsilon_0 > 0$, (which may depend on constants in our assumptions but not on the choice of $\mathfrak{A}(N)$), such that for each $0 < \varepsilon < \varepsilon_0$ there exists $N_0 \in \mathbb{Z}_{\geq 1}$, and for all $N > N_0$ we have*

- $\text{Re } S'(x + i\varepsilon) < -1$ for all $-1 - \varepsilon^\delta < x < -1 + \varepsilon^\delta$;
- $\text{Re } S'(x + i\varepsilon) > 1$ for all $-\varepsilon^\delta < x < \varepsilon^\delta$.

Proof. We will prove only the second claim, as the first one is analogous. We will specify the exact value of ε_0 at the end of the proof, and for now let us just fix arbitrary $\varepsilon < \varepsilon_0 < 1$ and $\delta \in (0, 1)$. In addition, take a large positive real R . If R and N are large enough, then we can

¹²One can show that these curves do not intersect, i.e., that $S''(z)$ cannot vanish where $\text{Im } S'(z) = 0$, but we do not need this fact.

restrict the summation in the infinite principal value sum in (5.3) to $j \in \mathfrak{L}_T \cap [-RT, RT]$, so that

$$\left| \text{p.v.} \sum_{j \in \mathfrak{L}_T} \frac{1}{T} \frac{x - j/T}{y^2 + (x - j/T)^2} - \sum_{j \in \mathfrak{L}_T \cap [-RT, RT]} \frac{1}{T} \frac{x - j/T}{y^2 + (x - j/T)^2} \right| < 1, \quad y = \varepsilon.$$

In turn, the sum over $j \in \mathfrak{L}_T \cap [-RT, RT]$ is the Riemann sum for the corresponding integral, so for large N we have

$$\left| \text{p.v.} \sum_{j \in \mathfrak{L}_T} \frac{1}{T} \frac{x - j/T}{y^2 + (x - j/T)^2} - \int_{u \in [-R, -1] \cup [0, R]} \frac{(x - u) du}{y^2 + (x - u)^2} \right| < 2, \quad y = \varepsilon. \quad (5.7)$$

Let us now bound the sum over the configuration $\mathfrak{A}(N)$ in (5.3). For that we split this sum into three parts:

$$\begin{aligned} \sum_{i \in \mathfrak{A}(N)} \frac{1}{T} \frac{x - i/T}{y^2 + (x - i/T)^2} &= \sum_{i \in \mathfrak{A}(N) \cap [-RT, RT]} \frac{1}{T} \frac{x - i/T}{y^2 + (x - i/T)^2} \\ &+ \sum_{i \in \mathfrak{A}(N) \setminus [-RT, RT]} \frac{1}{T} \left(\frac{x - i/T}{y^2 + (x - i/T)^2} + \frac{T}{i} \right) - \sum_{i \in \mathfrak{A}(N) \setminus [-RT, RT]} \frac{1}{i}. \end{aligned} \quad (5.8)$$

The third sum in (5.8) is bounded due to Assumption 2. For the second sum, observe that

$$\frac{1}{T} \left(\frac{x - i/T}{y^2 + (x - i/T)^2} + \frac{T}{i} \right) = \frac{1}{T} \frac{y^2 + x^2 - (i/T)x}{(i/T)(y^2 + (x - i/T)^2)},$$

that is, the second sum over i converges absolutely. Moreover, we can estimate as $N \rightarrow \infty$:

$$\sum_{i \in \mathfrak{A}(N) \setminus [-RT, RT]} \left| \frac{1}{T} \left(\frac{x - i/T}{y^2 + (x - i/T)^2} + \frac{T}{i} \right) \right| \leq \sum_{i \in \mathbb{Z} \setminus [-RT, RT]} \left| \frac{1}{T} \frac{y^2 + x^2 - (i/T)x}{(i/T)(y^2 + (x - i/T)^2)} \right|,$$

and the right-hand side is the Riemann sum for the integral

$$\int_{\mathbb{R} \setminus [-R, R]} \left| \frac{y^2 + x^2 - ux}{u(y^2 + (x - u)^2)} \right| du,$$

which is uniformly bounded for (x, y) in our segment (where x is around 0). Thus, the second sum in (5.8) is uniformly bounded by a constant independent of ε .

Finally, for the first sum in (5.8) we use Assumption 1 and approximate sums by integrals similarly to the proof of Lemma 5.2. To get a lower bound, first throw away all nonnegative summands in this sum, and write for the remaining ones:

$$\sum_{\substack{i \in \mathfrak{A}(N) \cap [-RT, RT] \\ i/T > x}} \frac{1}{T} \frac{x - i/T}{y^2 + (x - i/T)^2} > \frac{1 + \rho^\bullet}{2} \int_x^R \frac{(x - u) du}{y^2 + (x - u)^2}, \quad (5.9)$$

where N is sufficiently large.

Combining all the estimates, we obtain the following bound. For each $\varepsilon > 0$ there exists N_0 such that for all $N > N_0$ we have

$$\begin{aligned} \operatorname{Re} S'(x + iy) &> \frac{1 + \rho^\bullet}{2} \int_x^R \frac{(x-u)du}{y^2 + (x-u)^2} - \int_{u \in [-R, -1] \cup [0, R]} \frac{(x-u)du}{y^2 + (x-u)^2} \\ &\quad - \log(\beta^{-1} - 1) + \text{“error”}, \end{aligned} \quad (5.10)$$

where “error” is uniform in (x, y) in our segment and is independent of ε . Observe that

$$\int_x^R \frac{(x-u)du}{y^2 + (x-u)^2} = \int_0^{R-x} \frac{-v dv}{v^2 + y^2} > \int_0^{R+1} \frac{-v dv}{v^2 + y^2},$$

and

$$-\int_0^R \frac{(x-u)du}{y^2 + (x-u)^2} = \int_{-x}^{R-x} \frac{v dv}{v^2 + y^2} > \int_{-x}^{R-1} \frac{v dv}{v^2 + y^2}.$$

At the same time,

$$-\int_{-R}^{-1} \frac{(x-u)du}{y^2 + (x-u)^2} = -\frac{1}{2} \log \left(\frac{(R+x)^2 + y^2}{(1+x)^2 + y^2} \right)$$

can be bounded by an absolute constant since both x and y are close to zero. Thus, we can write

$$\begin{aligned} &\frac{1 + \rho^\bullet}{2} \int_x^R \frac{(x-u)du}{y^2 + (x-u)^2} - \int_{u \in [-R, -1] \cup [0, R]} \frac{(x-u)du}{y^2 + (x-u)^2} \\ &> \frac{1 + \rho^\bullet}{2} \int_0^{R+1} \frac{-v dv}{v^2 + y^2} + \int_{-x}^{R-1} \frac{v dv}{v^2 + y^2} + C \\ &= -\frac{1 + \rho^\bullet}{4} \log \left(\frac{(R+1)^2}{y^2} + 1 \right) + \frac{1}{2} \log \left(\frac{(R-1)^2 + y^2}{x^2 + y^2} \right) + C. \end{aligned}$$

Here and below in this proof C stands for some real constant which is uniform in x, y and does not depend on ε but may depend on R (but we fixed large R once and for all in the beginning of the proof). The value of C can change from line to line. Since $y = \varepsilon$ is small, we have

$$-\frac{1 + \rho^\bullet}{4} \log \left(\frac{(R+1)^2}{y^2} + 1 \right) = \frac{1 + \rho^\bullet}{2} \log y + C + O(\varepsilon^2).$$

We also have

$$\frac{1}{2} \log \left(\frac{(R-1)^2 + y^2}{x^2 + y^2} \right) = \frac{1}{2} \log (y^2 + (R-1)^2) - \log \sqrt{x^2 + y^2} > C - \delta \log y$$

because $x^2 + y^2 < \varepsilon^{2\delta} + \varepsilon^2$, which behaves as $\varepsilon^{2\delta}(1 + o(1)) = y^{2\delta}(1 + o(1))$. When δ is close enough to 1,

$$\frac{1 + \rho^\bullet}{2} \log y - \delta \log y$$

tends to $+\infty$ as $y = \varepsilon \rightarrow 0$, and we are done. \square

Proof of Proposition 2.6. Fix $\varepsilon > 0$ and $N_0 \in \mathbb{Z}_{\geq 1}$ depending on ε such that Lemmas 5.2, 5.4 and 5.5 hold (recall that $\varepsilon < \varepsilon_0$, where ε_0 may depend on constants in our assumptions but not on the choice of $\mathfrak{A}(N)$). Consider the curve γ from Lemma 5.4. This is a continuous curve on which $\operatorname{Im} S'(z) = 0$. Furthermore, Lemma 5.5 guarantees that $\operatorname{Re} S'(z)$ has distinct signs at the endpoints of γ . Since $\operatorname{Re} S'(z)$ is a continuous function on γ , we conclude that there exists $z_c \in \gamma$ for which $\operatorname{Re} S'(z_c) = 0$, and so $S'(z_c) = 0$ (as S' depends on N , so does z_c , and this statement is valid for all $N > N_0$). Lemma 5.1 then implies that there are no other critical points in the upper half plane, and therefore z_c is the desired unique one. As the compact set \mathcal{Z} capturing z_c take $\{x + iy \in \mathbb{C} : y \geq \varepsilon, \sqrt{x^2 + y^2} \leq \varepsilon^{-1}\}$. \square

6. ASYMPTOTICS OF THE KERNEL: PROOFS OF THEOREMS 2.7 AND 2.10 TO 2.12

In this section, based on the existence of nonreal critical points afforded by Proposition 2.6, we establish the approximation of the correlation kernel (2.2) of the noncolliding Bernoulli random walk by the extended sine kernel, and also the corresponding bulk limit theorems. That is, here we prove the remaining statements from Section 2.3.

6.1. Behavior of $\operatorname{Im} S(z)$. We aim to describe the steepest descent contours for $\operatorname{Re} S(z)$. For that let us look at the behavior of $\operatorname{Im} S(z)$ close to the boundary of the upper half plane \mathbb{H} , i.e., for z close to the real line or for $|z|$ large. Recall that we defined $S(z)$ in Section 4.2 so that it is holomorphic in \mathbb{H} and extends to the real axis except the singularities at $\iota_T \Delta \alpha$ (all other logarithmic singularities belonging to $\frac{1}{T}\mathbb{Z}$ are removable).

We begin by considering the behavior of $\operatorname{Im} S(z)$ for large $|z|$.

Proposition 6.1. *Fix $c \in \mathbb{R}$ and $N \in \mathbb{Z}_{\geq 1}$. There exists $\zeta_0 > 0$ such that for any $0 < \zeta < \zeta_0$ there is exactly one solution to the equation $\operatorname{Im} S(\zeta^{-1}e^{i\theta}) = c$ in $\theta \in [0, \pi]$, and the solution belongs to $(0, \pi)$. Hence there is exactly one contour escaping to infinity along which $\operatorname{Im} S(z) = c$. Moreover, along this contour $\operatorname{Re} S(z)$ decays to $-\infty$.*

Proof. Using (4.7) we have for $z = \zeta^{-1}e^{i\theta}$, where $\theta \in (0, \pi)$ and $\zeta > 0$ is small:

$$\operatorname{Im} S(\zeta^{-1}e^{i\theta}) = \pi \zeta^{-1} \cos \theta - \log(\beta^{-1} - 1) \zeta^{-1} \sin \theta + o(\zeta^{-1}), \quad (6.1)$$

where the remainder, as well as its θ -derivative are uniformly (in θ) small as $\zeta \rightarrow 0$. This implies that any solution to $\operatorname{Im} S(\zeta^{-1}e^{i\theta}) = c$ must be close to the solution to

$$\pi \cos \theta - \log(\beta^{-1} - 1) \sin \theta = 0.$$

Since the coefficient by $\cos \theta$ is always positive, this can be rewritten as $\cos(\theta + \vartheta) = 0$, where $\vartheta \in (-\pi/2, \pi/2)$ is an angle with $\sin \vartheta = \frac{\log(\beta^{-1}-1)}{\sqrt{\pi^2 + \log(\beta^{-1}-1)^2}}$. Since $\theta + \vartheta \in (-\pi/2, 3\pi/2)$, this equation has a unique solution $\theta = \pi/2 - \vartheta \in (0, \pi)$. The derivative of (6.1) at $\theta = \pi/2 - \vartheta$ is a nonzero constant times ζ^{-1} , and hence by possibly decreasing ζ_0 one can make sure that the solution to $\operatorname{Im} S(\zeta^{-1}e^{i\theta}) = c$ is unique.

Finally, along the direction $\theta = \vartheta$ the real part of $S(\zeta^{-1}e^{i(\pi/2-\vartheta)})$ gets the main contribution from the real part of (6.1) which is equal to

$$\zeta^{-1}(-\pi \sin(\pi/2 - \vartheta) - \log(\beta^{-1} - 1) \cos(\pi/2 - \vartheta)) = -\zeta^{-1} \sqrt{\pi^2 + \log(\beta^{-1} - 1)^2} \rightarrow -\infty,$$

as desired. \square

Let us now consider the behavior of $\text{Im } S(z)$ close to the real line, and define

$$\begin{aligned} b_0^r &= \min(\mathbb{Z}_{\geq 0} \setminus \mathfrak{A}), & b_0^\ell &= \max(\mathfrak{A} \cap \{-T+1, \dots, -1\}), \\ b_{-1}^\ell &= \max(\mathbb{Z}_{\leq -T} \setminus \mathfrak{A}), & b_{-1}^r &= \min(\mathfrak{A} \cap \{-T+1, \dots, -1\}). \end{aligned}$$

Clearly,

$$b_{-1}^\ell \leq -T < -T+1 \leq b_{-1}^r \leq b_0^\ell \leq -1 < 0 \leq b_0^r.$$

Lemma 6.2. *For $x \in \mathbb{R}$, $x \notin \mathfrak{l}_T \Delta \mathfrak{a}$, the function $\text{Im } S(x)$ is piecewise constant, making jumps at points $\in \mathfrak{l}_T \Delta \mathfrak{a}$. It weakly increases for $x \in (-\infty, b_{-1}^r/T) \cup (b_0^\ell/T, +\infty)$, and weakly decreases for $x \in (b_{-1}^\ell/T, b_0^r/T)$. See Figure 10 for an example.*

Proof. This is straightforward from the definition of $S(z)$ in Section 4.2 and the observation that $\text{Im}(\log_{\mathbb{H}}(x)) = \pi \mathbf{1}_{x < 0}$, where $x \in \mathbb{R} \setminus \{0\}$. \square

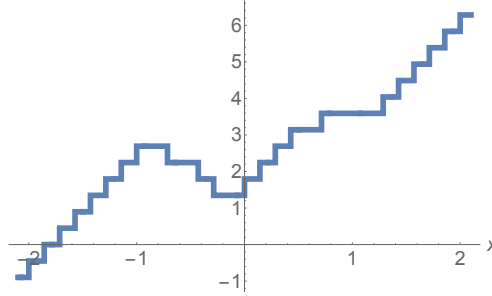


FIGURE 10. Staircase-type plot of $\text{Im } S(x)$ for $x \in \mathbb{R}$, with parameters as in Figure 8. The singularities leading to the down steps are $\{-\frac{6}{7}, \dots, -\frac{1}{7}\} \cap \{-\frac{5}{7}, -\frac{3}{7}, -\frac{2}{7}, \frac{4}{7}, \frac{6}{7}, 1, \frac{8}{7}\}$.

Denote the staircase-type function $\text{Im } S(x + i0)$, $x \in \mathbb{R}$, by $\text{Im}_0 S(x)$ (see Figure 10). By a slight abuse of notation, let us also by $\text{Im}_0 S$ denote the continuous piecewise linear curve in the plane obtained from the plot of $\text{Im}_0 S(x)$ by adding vertical segments at the discontinuities.

Lemma 6.3. *There exists $C > 0$ and $\zeta_0 > 0$ such that for all $0 < \zeta < \zeta_0$ the plot of the (smooth) function $x \mapsto \text{Im } S(x + i\zeta)$ belongs to a $C\zeta^{\frac{1}{2}}$ -neighborhood of the curve $\text{Im}_0 S$.*

Proof. For $\zeta > 0$ we have

$$\text{Im} \log_{\mathbb{H}}(x + i\zeta) = \text{Im} \log(x + i\zeta) = \arg(x + i\zeta),$$

where $\arg \in (-\pi, \pi]$ is the standard argument. Applying this to $\text{Im } S(x + i\zeta)$ we see that all the logarithms in (4.8) contribute shifts (with coefficients ± 1) of one and the same function, $x \mapsto \arg(x + i\zeta)/T$ (for $\log(\sin(\pi Tz))$ we use (4.5) because we are strictly in the upper half plane). Observe that this function is strictly decreasing in x , and

$$\arg(x + i\zeta) = \pi \mathbf{1}_{x < 0} + O(\zeta^{\frac{1}{2}}), \quad |x| > \zeta^{\frac{1}{2}}, \quad (6.2)$$

cf. Figure 11. The contribution of the term $-z \log(\beta^{-1} - 1)$ is of order ζ and can be ignored. \square

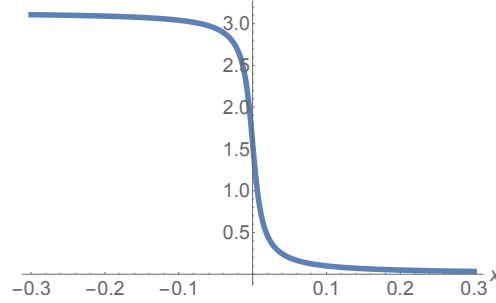


FIGURE 11. Plot of the function $\arg(x + i\zeta)$ for $\zeta = 0.01$.

Recall the critical point z_c afforded by Proposition 2.6.

Proposition 6.4. *Fix $N \in \mathbb{Z}_{\geq 1}$. Under Assumptions 1 and 2 there exist $\zeta_0 > 0$ and $B > 0$ (which may depend on constants in our assumptions but not on the choice of $\mathfrak{A}(N)$) such that for all $0 < \zeta < \zeta_0$ the equation $\operatorname{Im} S(x + i\zeta) = \operatorname{Im} S(z_c)$ has exactly three solutions in $x \in \mathbb{R}$ (counted with multiplicities) belonging to the intervals*

$$(-B, b_{-1}^r/T), \quad (b_{-1}^\ell/T, b_0^r/T), \quad \text{and} \quad (b_0^\ell/T, B), \quad (6.3)$$

respectively (note that the intervals overlap).

Proof. By Proposition 2.6, there is a unique critical point z_c of S in the upper half plane, and there are four contours $\{z: \operatorname{Im} S(z) = \operatorname{Im} S(z_c)\}$ emanating from the critical point. Proposition 6.1 guarantees that exactly one of these contours escapes to infinity, so the other three contours must end at the real line. This implies that there are at least three solutions to $\operatorname{Im} S(x + i\zeta) = \operatorname{Im} S(z_c)$.

Denote by L the horizontal line at height $\operatorname{Im} S(z_c)$. First, assume that $\operatorname{Im} S(z_c) \notin \frac{\pi}{T}\mathbb{Z}$, i.e., L intersects vertical and not horizontal segments of the staircase-type curve $\operatorname{Im}_0 S$. In this case solutions to the staircase-type equation $\operatorname{Im}_0 S(x) = \operatorname{Im} S(z_c)$ are in one-to-one correspondence with solutions to $\operatorname{Im} S(x + i\zeta) = \operatorname{Im} S(z_c)$ for sufficiently small ζ . Indeed, by Lemma 6.3 all possible solutions to $\operatorname{Im} S(x + i\zeta) = \operatorname{Im} S(z_c)$ corresponding to one solution to the staircase-type equation belong to a neighborhood of a singularity of one of the logarithms in (4.8). This logarithm dominates in this neighborhood for small enough ζ , and thus the smooth function $\operatorname{Im} S(x + i\zeta)$ is strictly monotone there. Thus by Lemma 6.2 there are at most three solutions to $\operatorname{Im} S(x + i\zeta) = \operatorname{Im} S(z_c)$, and hence exactly three solutions.

Now consider the case $\operatorname{Im} S(z_c) \in \frac{\pi}{T}\mathbb{Z}$. In this case L intersects $\operatorname{Im}_0 S$ by whole horizontal segments between consecutive singularities of S (which are $l_T \Delta \mathfrak{a}$). Each solution x_ζ to $\operatorname{Im} S(x + i\zeta) = \operatorname{Im} S(z_c)$ is close to a real critical point of S (i.e., where $S'(x_0) = 0$). Indeed, this is because $\operatorname{Im} S(z) = \operatorname{Im} S(z_c)$ for z on the corresponding segment of the real line and also at $z = x_\zeta + i\zeta$ which is close to the interior of this segment. Since the nonreal

critical point z_c exists by Proposition 2.6, the argument in the proof of Lemma 5.1 implies that each segment of the real line between two consecutive singularities of S contains one such critical point x_0 , except the segments $(b_{-1}^r/T, b_{-1}^\ell)$ and $(b_0^r/T, b_0^\ell)$ which have no critical points. Lemma 6.2 then implies that there can be at most three solutions to $\operatorname{Im} S(x + i\zeta) = \operatorname{Im} S(z_c)$, and hence exactly three solutions.

The claim about the segments (6.3) is straightforward for $B = +\infty$. However, we can take B finite because to get three solutions to $\operatorname{Im} S(x + i\zeta) = \operatorname{Im} S(z_c)$, the line L must intersect the middle (decreasing) part of $\operatorname{Im}_0 S$. If L intersects an increasing part far away, then the slope of this increasing part is too small, but one readily sees that this contradicts Assumption 1. \square

6.2. Moving the contours: proof of Theorem 2.7. Let us fix $N_0 \in \mathbb{Z}_{\geq 1}$ and the compact set $\mathcal{Z} \subset \mathbb{H}$ afforded by Proposition 2.6 such that for all $N > N_0$ there exists a unique critical point $z_c = z_c(N)$ of S belonging to \mathcal{Z} . There are four half-contours emanating from z_c on which $\operatorname{Im} S(z) = \operatorname{Im} S(z_c)$. By Propositions 6.1 and 6.4, one of these half-contours escapes to infinity (and on it $\operatorname{Re} S(z)$ decreases), while three other half-contours end at the real line at three points belonging to the intervals $(-B, b_{-1}^r/T)$, $(b_{-1}^\ell/T, b_0^r/T)$, and $(b_0^\ell/T, B)$, respectively. An example is shown in Figure 12. The configuration of the corresponding contours in the lower half plane (with a suitable choice of branches of logarithms, cf. Section 4.2) is obtained by reflection with respect to the real line.

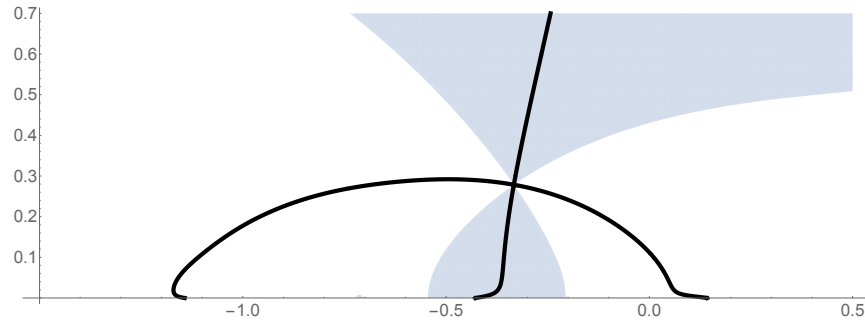


FIGURE 12. The steepest descent/ascent contours $\{z: \operatorname{Im} S(z) = \operatorname{Im} S(z_c)\}$ for the function $\operatorname{Re} S(z)$ in \mathbb{H} . Regions where $\operatorname{Re} S(z) < \operatorname{Re} S(z_c)$ are shaded. The parameters \mathfrak{A} and T are as in Figure 8, and $\beta = 0.4$.

Fix small $\epsilon < \frac{1}{4} \operatorname{distance}(\mathcal{Z}, \mathbb{R})$. Denote by $\mathcal{C}_z^\epsilon = \mathcal{C}_z^\epsilon(N)$ the contour which at distance ϵ away from \mathbb{R} coincides with the union of two half-contours $\{z: \operatorname{Im} S(z) = \operatorname{Im} S(z_c)\}$ on which $\operatorname{Re} S(z) < \operatorname{Re} S(z_c)$, plus a union of contours in the lower half plane obtained by reflection with respect to \mathbb{R} . We want \mathcal{C}_z^ϵ to cross \mathbb{R} strictly between -1 and 0 , and this is achieved as follows. When \mathcal{C}_z^ϵ reaches one of the horizontal lines $\operatorname{Im} z = \pm i\epsilon$, it first goes horizontally, and then has a vertical part which crosses \mathbb{R} at a point of the form $(k+1/2)/T$ for some $k \in \mathbb{Z}$. The contour \mathcal{C}_z^ϵ passes through infinity and is oriented upwards. By $\mathcal{C}_w^\epsilon = \mathcal{C}_w^\epsilon(N)$ denote the closed positively oriented contour which at distance ϵ away from the real line coincides with the steepest descent contour $\{z: \operatorname{Im} S(z) = \operatorname{Im} S(z_c)\}$ on which

$\operatorname{Re} S(w) > \operatorname{Re} S(z_c)$, encircles all the points $\{-1 + T^{-1}(\Delta x - \Delta t), \dots, T^{-1}(\Delta x - 1), T^{-1}\Delta x\}$, and crosses \mathbb{R} at a point of the form $(k + 1/2)/T$ (using horizontal and vertical segments similarly to \mathcal{C}_z^ϵ). See Figure 13 for an illustration.

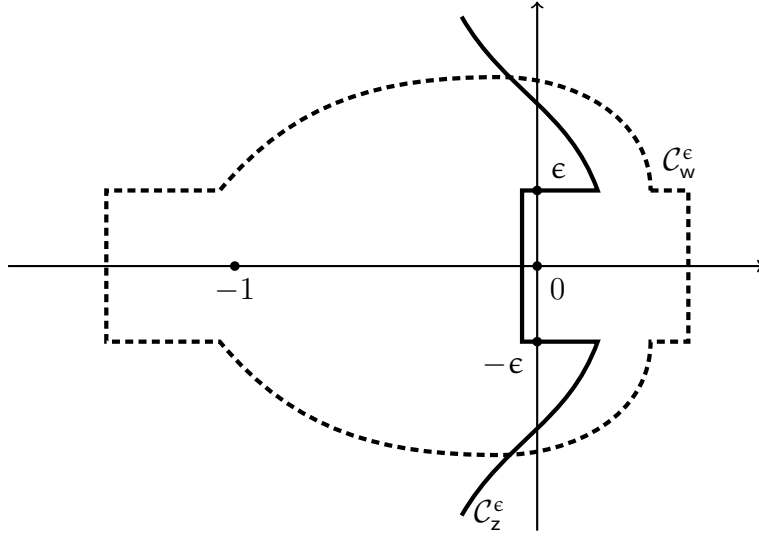
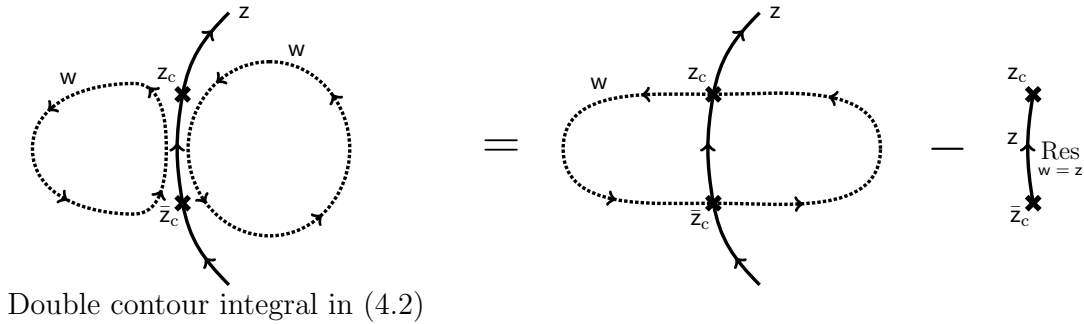


FIGURE 13. Contours \mathcal{C}_z^ϵ and \mathcal{C}_w^ϵ .

Proposition 6.5. *With the above definitions and conventions, for any $N > N_0$ the kernel (4.2) can be written as*

$$\begin{aligned}
 K_{\vec{a};\beta}^{\text{Bernoulli}}(T + \Delta t, \Delta x; T, 0) &= K_{z_c/(z_c+1)}(T + \Delta t, \Delta x; T, 0) + \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_z^\epsilon} dz \oint_{\mathcal{C}_w^\epsilon} dw \frac{1}{w - z} \\
 &\times \frac{(T + \Delta t)! \cdot T}{(T - 1)!} \frac{(Tz + 1)_{T-1}}{(Tw - \Delta x)_{T+\Delta t+1}} \frac{\sin(\pi Tw)}{\sin(\pi Tz)} \left(\frac{1 - \beta}{\beta} \right)^{T(w-z)} \prod_{r=1}^N \frac{Tz - a_r}{Tw - a_r} + O(T^{-1}).
 \end{aligned} \tag{6.4}$$



Double contour integral in (4.2)

FIGURE 14. Unifying two circles into a single w contour.

Proof. All poles of the integrand in (4.2) are on the real line. This integrand has no poles at $z \in \{b_{-1}^\ell/T, \dots, b_0^r/T\}$, and thus we can drag the point of intersection of the z contour with the real line to the desired location dictated by the contour \mathcal{C}_z^ϵ . Hence we can deform the whole z contour to coincide with \mathcal{C}_z^ϵ without crossing any poles. Next, let us unite the two circles comprising the w contour in (4.2) into the contour \mathcal{C}_w^ϵ intersecting with \mathcal{C}_z^ϵ at the critical points z_c and \bar{z}_c . This leads to an additional integral of the residue at $w = z$ over the arc of \mathcal{C}_z^ϵ from \bar{z}_c to z_c crossing the real line between -1 and 0 , see Figure 14. This deformation of the w contour does not cross any other w poles of the integrand.¹³

The expression coming from the residue of the integrand at $w = z$ behaves as

$$\begin{aligned} & -\frac{1}{2\pi i} \frac{(T + \Delta t)! \cdot T}{(T-1)!} \frac{(T\mathbf{w} + 1)_{T-1}}{(T\mathbf{w} - \Delta x)_{T+\Delta t+1}} \\ & = -\frac{1}{2\pi i} (1 + O(T^{-1})) T^{\Delta t+2} \frac{\Gamma(T\mathbf{w} + T)\Gamma(T\mathbf{w} - \Delta x)}{\Gamma(T\mathbf{w} + 1)\Gamma(T\mathbf{w} - \Delta x + T + \Delta t + 1)} \\ & = -\frac{1}{2\pi i} (1 + O(T^{-1})) \mathbf{w}^{-\Delta x-1} (1 + \mathbf{w})^{-\Delta t + \Delta x-1}, \quad (6.5) \end{aligned}$$

where we used (3.13), and the asymptotic expression is valid for all $\mathbf{w} \in \mathcal{C}_w^\epsilon$ (for real $\mathbf{w} < -1$ one should apply (3.12) to all four gamma functions before using (3.13)).

The right-hand side of (6.5) above has singularities at $\mathbf{w} = -1$ and $\mathbf{w} = 0$, and the arc of the contour \mathcal{C}_z^ϵ between the critical points \bar{z}_c and z_c crosses $(-1, 0)$. The integral of the error $O(T^{-1})$ in (6.5) over the arc from \bar{z}_c to z_c is bounded by $O(|z_c|/T)$, which is $O(1/T)$ because z_c belongs to a compact set \mathcal{Z} .

Let us now identify the extended sine kernel (2.3) in the remaining terms outside the double contour integral over \mathcal{C}_z^ϵ and \mathcal{C}_w^ϵ . Observe that for $\Delta t \geq 0$ we have

$$\text{Res}_{\mathbf{w}=0} \left(\mathbf{w}^{-\Delta x-1} (1 + \mathbf{w})^{-\Delta t + \Delta x-1} \right) = -(-1)^{\Delta x+1} \binom{\Delta t}{\Delta x} \mathbf{1}_{\Delta x \geq 0}.$$

By dragging the integration arc through 0 for $\Delta t > 0$ we obtain

$$\begin{aligned} & \mathbf{1}_{\Delta x \geq 0} \mathbf{1}_{\Delta t > 0} (-1)^{\Delta x+1} \binom{\Delta t}{\Delta x} - \frac{1}{2\pi i} \int_{z_c}^{\bar{z}_c} \mathbf{w}^{-\Delta x-1} (1 + \mathbf{w})^{-\Delta t + \Delta x-1} d\mathbf{w} \\ & = -\frac{1}{2\pi i} \int_{z_c}^{\bar{z}_c} \mathbf{w}^{-\Delta x-1} (1 + \mathbf{w})^{-\Delta t + \Delta x-1} d\mathbf{w}, \end{aligned}$$

where in the right-hand side the arc crosses $(-1, 0)$ for $\Delta t \leq 0$ and $(0, +\infty)$ for $\Delta t > 0$. Changing the variables in the right-hand side as $\mathbf{w} = \frac{z}{1-z}$ (so $z = \frac{\mathbf{w}}{1+\mathbf{w}}$) turns the above integral into $K_{z_c/(z_c+1)}(T + \Delta t, \Delta x; T, 0)$ (2.3), as desired. \square

¹³The desire that these deformations do not cross any real poles is the reason for defining the contours \mathcal{C}_z^ϵ and \mathcal{C}_w^ϵ .

To complete the proof of Theorem 2.7 it remains to show that the double contour integral in (6.4) is negligible as $N \rightarrow +\infty$. It has the form (cf. (4.3), (4.8))

$$\frac{1}{(2\pi i)^2} \int_{\mathcal{C}_z^\epsilon} dz \oint_{\mathcal{C}_w^\epsilon} dw \frac{1}{w - z} \exp \left\{ T(S(z) - S(w)) \right\} \frac{(T + \Delta t)! \cdot T}{(T - 1)!} \frac{(T\mathbf{w} + 1)_{T-1}}{(T\mathbf{w} - \Delta x)_{T+\Delta t+1}}. \quad (6.6)$$

First, recall that the parts in (6.6) outside the exponent are bounded by a constant depending on $\Delta x, \Delta t$ due to (6.5) and the fact that \mathbf{w} belongs to the contour \mathcal{C}_w^ϵ whose length is bounded uniformly in N . We establish this fact as Lemma 6.8 below.

Now, for z and \mathbf{w} in a fixed small neighborhood of the critical point $z_c \in \mathcal{Z}$ which is bounded away from \mathbb{R} we can Taylor expand the function $S(z)$. This leads to a convergent integral times $T^{-\frac{1}{2}}$, which goes to zero. This is a standard part of the steepest descent analysis, and we refer to, e.g., [Oko02, Section 3] for details.

Next, when z is outside of this neighborhood of z_c , the real part of $S(z)$ strictly decays along the steepest descent contour $\text{Im } S(z) = \text{Im } S(z_c)$. Since the integration contour \mathcal{C}_z^ϵ differs from the steepest descent one close to the real line, we need to show that the value of $\text{Re } S(z)$ does not increase much on the straight segments close to \mathbb{R} . Similarly, we need to show that the value of $\text{Re } S(\mathbf{w})$ does not decrease much for \mathbf{w} on the straight segments close to \mathbb{R} .

Lemma 6.6. 1. *For any fixed $k \in \mathbb{Z}$ and $y > 0$ we have*

$$\left| \frac{\partial}{\partial y} \text{Re } S\left(\frac{k + 1/2}{T} + iy\right) \right| \leq \pi.$$

2. *For any fixed $\epsilon > 0$ we have*

$$\limsup_{N \rightarrow +\infty} \left| \frac{\partial}{\partial x} \text{Re } S(x + i\epsilon) \right| \leq \text{const} \cdot \log(\epsilon^{-1}),$$

where the constant is uniform in x on bounded segments.

Proof. Using (4.5) we see that, apart from the linear term $-(\text{Re } z) \log(\beta^{-1} - 1)$, the function $\text{Re } S(z)$ is an infinite linear combination (with coefficients ± 1) of shifts of $\frac{1}{T} \log |z|$. We have for any $j \in \mathbb{Z}$:

$$\left| \frac{\partial}{\partial y} \frac{1}{T} \log \left| \frac{k + 1/2}{T} + iy + \frac{j}{T} \right| \right| = \frac{4T|y|}{(2k + 2j + 1)^2 + 4T^2y^2}.$$

Thus, the derivative of $\text{Re } S$ in the first claim can be bounded in the absolute value by

$$\sum_{j \in \mathbb{Z}} \frac{4T|y|}{(2j + 1)^2 + 4T^2y^2} = \pi \tanh(\pi T|y|) \leq \pi,$$

this is summed with the help of a partial fraction expansion and (2.10), and \tanh is bounded by one. This establishes the first claim.

Let us turn to the second claim. Assume that x is fixed. We have

$$\frac{\partial}{\partial x} \frac{1}{T} \log \left| x + i\epsilon + \frac{j}{T} \right| = \frac{Tx + j}{\epsilon^2 T^2 + (Tx + j)^2},$$

so

$$\begin{aligned} \frac{\partial}{\partial x} \operatorname{Re} \mathbf{S}(x + \mathbf{i}\epsilon) &= \sum_{r=1}^N \frac{Tx - a_r}{\epsilon^2 T^2 + (Tx - a_r)^2} + \sum_{i=1}^{T-1} \frac{Tx + i}{\epsilon^2 T^2 + (Tx + i)^2} \\ &\quad - \log(\beta^{-1} - 1) - \frac{\partial}{\partial x} \frac{1}{T} \log |\sin(\pi T(x + \mathbf{i}\epsilon))|. \end{aligned} \quad (6.7)$$

Fix sufficiently large $R > 0$. For the first sum in (6.7) with $|a_r| < RT$, and also for the second sum in (6.7) we use the fact that

$$\sum_{j=0}^{RT} \frac{j}{j^2 + \epsilon^2 T^2} \sim \int_0^R \frac{v}{v^2 + \epsilon^2} dv = \frac{1}{2} \log \left(1 + \frac{R^2}{\epsilon^2} \right).$$

Next, for $|a_r| > RT$ in the first sum in (6.7) we have

$$\frac{1}{Tx - a_r} - \frac{Tx - a_r}{\epsilon^2 T^2 + (Tx - a_r)^2} = \frac{T^2 \epsilon^2}{(Tx - a_r)(T^2 \epsilon^2 + (Tx - a_r)^2)},$$

and summing this over $|a_r| > RT$ has order ϵ^2/R^2 , which is bounded. Thus, the contribution from $|a_r| > RT$ in the first sum in (6.7) is the same as if the summands were just $1/(Tx - a_r)$. Observe that

$$\frac{1}{Tx - a_r} + \frac{1}{a_r} = \frac{Tx}{a_r(Tx - a_r)}$$

and the sum of these quantities over $|a_r| > RT$ with large R is bounded (recall that x is fixed, and here the bound on R may depend on x). Thus, the sum of the terms with $|a_r| > RT$ in the first sum in (6.7) has the same order as the sum of $1/a_r$ over $|a_r| > RT$, which is bounded by Assumption 2. Finally, for the last summand in (6.7) we have

$$\begin{aligned} \frac{\partial}{\partial x} \frac{1}{T} \log |\sin(\pi T(x + \mathbf{i}\epsilon))| &= \frac{\partial}{\partial x} \frac{1}{2T} \log \left(\frac{1}{2} (\cosh(2\pi T\epsilon) - \cos(2\pi Tx)) \right) \\ &= \frac{\pi \sin(2\pi Tx)}{\cosh(2\pi T\epsilon) - \cos(2\pi Tx)}. \end{aligned}$$

Because $\cosh(2\pi T\epsilon)$ grows exponentially in T , the above expression goes to zero (note that in the statement we first fix ϵ and let $N \rightarrow +\infty$, and then consider the behavior of the limit for small ϵ). This implies that

$$\left| \frac{\partial}{\partial x} \operatorname{Re} \mathbf{S}(x + \mathbf{i}\epsilon) \right| = O(\log(\epsilon^{-1})),$$

as desired. \square

Lemma 6.6 shows that the contribution to the double contour integral (6.6) from parts of the contours \mathcal{C}_z^ϵ and \mathcal{C}_w^ϵ , respectively, which are close to \mathbb{R} , is also negligible. Indeed, the vertical parts of the contours are of length 2ϵ (cf. Figure 13), and on them the change of $\operatorname{Re} \mathbf{S}(z)$ is of order ϵ by the first claim in Lemma 6.6. One also readily sees that the horizontal parts of the contours have length of order D/T , which gets smaller than ϵ for

large N , and thus we can use the second claim of Lemma 6.6. This implies that the whole double contour integral (6.6) is negligible as $N \rightarrow +\infty$, and completes the proof of Theorem 2.7.

6.3. Convergent initial data: proofs of Theorems 2.10 to 2.12. In addition to Assumptions 1 and 2, let

$$\lim_{N \rightarrow +\infty} \frac{1}{T(N)} \sum_{i=1}^N \delta_{a_i(N)/T(N)} = \mu_{loc}, \quad (6.8)$$

where μ_{loc} is a σ -finite measure, and the limit is understood according to Definition 2.9. By Assumption 1, μ_{loc} has a density (with respect to the Lebesgue measure) which is between ρ_\bullet and ρ^\bullet .

Let us also assume that the quantities (2.12) have a limit $\mathbf{d}(\mathbf{R}) = \lim_{N \rightarrow +\infty} \mathbf{d}_N(\mathbf{R})$. Then the meromorphic function $\mathbf{S}'(\mathbf{z})$ (2.6) has a limit as $N \rightarrow +\infty$:

Lemma 6.7. *Under the above assumptions and notation, we have $\lim_{N \rightarrow +\infty} \mathbf{S}'(\mathbf{z}) = \mathbf{S}'_*(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{H}$, where*

$$\begin{aligned} \mathbf{S}'_*(\mathbf{z}) = \int_{-\infty}^{\infty} \left(\frac{1}{\mathbf{z} - v} + \frac{\mathbf{1}_{|v| > \mathbf{R}}}{v} \right) \mu_{loc}(dv) - \mathbf{d}(\mathbf{R}) \\ + \log(\mathbf{z} + 1) - \log \mathbf{z} + \mathbf{i}\pi - \log(\beta^{-1} - 1), \end{aligned} \quad (6.9)$$

and $\mathbf{R} > 0$ is arbitrary (the limit does not depend on \mathbf{R}). The convergence is uniform in \mathbf{z} belonging to compact subsets of \mathbb{H} .

Proof. Fix $\mathbf{z} \in \mathbb{H}$. Let us use formula (2.9) for $\mathbf{S}'(\mathbf{z})$. First, we have

$$-\pi \cot(\pi T\mathbf{z}) = -\mathbf{i}\pi \frac{e^{\mathbf{i}\pi T\mathbf{z}} + e^{-\mathbf{i}\pi T\mathbf{z}}}{e^{\mathbf{i}\pi T\mathbf{z}} - e^{-\mathbf{i}\pi T\mathbf{z}}} \rightarrow \mathbf{i}\pi,$$

because $e^{-\mathbf{i}\pi T\mathbf{z}}$ dominates for $\mathbf{z} \in \mathbb{H}$.

Next, the sum over $i = 1, \dots, T-1$ approximates the corresponding Riemann integral:

$$\sum_{i=1}^{T-1} \frac{1}{T\mathbf{z} + i} = \frac{1}{T} \sum_{i=1}^{T-1} \frac{1}{\mathbf{z} + i/T} \rightarrow \int_0^1 \frac{dv}{\mathbf{z} + v} = \log(\mathbf{z} + 1) - \log \mathbf{z},$$

and the convergence is uniform over \mathbf{z} in compact subsets of \mathbb{H} .

Finally, recall the definition of the atomic measure (5.4), and note that (6.8) means that the measures $\mathcal{M}_T[\mathbf{a}]$ vaguely converge to μ_{loc} . The remaining N -dependent part of (2.9) can be written as

$$\frac{1}{T} \sum_{r=1}^N \frac{1}{\mathbf{z} - a_r/T} = \int_{-\infty}^{+\infty} \frac{\mathcal{M}_T[\mathbf{a}](dv)}{\mathbf{z} - v}.$$

Since the function $1/(\mathbf{z} - v)$ in v does not have compact support, and its integral with respect to the Lebesgue measure diverges at infinity, one cannot directly apply (6.8) to the

integral above. Here we need a regularization afforded by the convergence of the constants $d_N(R)$ (2.12). Namely, take any $R > 0$ and write

$$\int_{-\infty}^{+\infty} \frac{\mathcal{M}_T[\mathbf{a}](dv)}{z-v} = \int_{-\infty}^{+\infty} \left(\frac{1}{z-v} + \frac{\mathbf{1}_{|v|>R}}{v} \right) \mathcal{M}_T[\mathbf{a}](dv) - d_N(R) \quad (6.10)$$

(this expression does not depend on R). Now the function under the integral decays as v^{-2} at infinity, and so is Lebesgue integrable. Since the density of μ_{loc} is bounded, by restricting the integration to $[-M, M]$ for large M and applying (6.8), we conclude that the integral converges to the corresponding integral with respect to μ_{loc} . The uniformity of each convergence above is evident, so this completes the proof. \square

The next statement was employed in Section 6.2 above, but since its proof is similar to the one of Lemma 6.7, we give it here. Lemma 6.8 is not involved in the proofs of Theorems 2.10 to 2.12 directly.

Lemma 6.8. *Under the assumptions of Theorem 2.7, the length of the contour \mathcal{C}_w^ϵ defined in Section 6.2 is bounded uniformly in N .*

Proof. Assume the contrary. Then there exists a subsequence $\{N_k\}$ along which the local measures converge in the sense of (6.8) (consider measures of segments with rational end-points and choose a diagonal subsequence), the constants $d_{N_k}(R)$ converge to $d(R)$, while the length of the contour \mathcal{C}_w^ϵ grows to infinity. To simplify notation, let us use the sequence $\{N\}$ instead of $\{N_k\}$ in the rest of the proof.

Let us show that there exist constants c_N such that there exists $\lim_{N \rightarrow +\infty} (S(z) - c_N)$, call it $S_*(z)$, uniformly in z in bounded subsets of \mathbb{H} . Moreover, S_* is holomorphic in \mathbb{H} , and its derivative is given by (6.9).

Fix $z \in \mathbb{H}$. Recall that $S(z)$ is given by (4.8), and let us consider the five summands in that formula separately. First, observe that

$$-\frac{1}{T} \log_{\mathbb{H}}(\sin(\pi Tz)) + \frac{2\pi i}{T} \left[\frac{1}{2} \operatorname{Re}(Tz) + \frac{1}{2} \right] - z \log(\beta^{-1} - 1) \rightarrow i\pi z - z \log(\beta^{-1} - 1)$$

due to (4.6), (4.7). Next, the second sum in (4.8) approximates a convergent Riemann integral:

$$\frac{1}{T} \sum_{i=1}^{T-1} \log_{\mathbb{H}} \left(z + \frac{i}{T} \right) \rightarrow \int_0^1 \log(z+v) dv = (z+1) \log(z+1) - z \log z - 1$$

(in \mathbb{H} the branches \log and $\log_{\mathbb{H}}$ coincide). The first sum in (4.8) can be rewritten as

$$\begin{aligned} \frac{1}{T} \sum_{r=1}^N \log_{\mathbb{H}} \left(z - \frac{a_r}{T} \right) &= \int_{-\infty}^{+\infty} \log(z-v) \mathcal{M}_T[\mathbf{a}](dv) \\ &= \int_{-\infty}^{+\infty} \left(\log(z-v) - \log(i-v) + (z-i) \frac{\mathbf{1}_{|v|>R}}{v} \right) \mathcal{M}_T[\mathbf{a}](dv) \\ &\quad + \int_{-\infty}^{+\infty} \log(i-v) \mathcal{M}_T[\mathbf{a}](dv) - (z-i) d_N(R). \end{aligned}$$

The integrand $\log(z - v) - \log(i - v) + (z - i) \frac{\mathbf{1}_{|v| > R}}{v}$ decays as v^{-2} at infinity, so the first integral in the right-hand side converges as $N \rightarrow +\infty$ to the same integral over μ_{loc} . The second integral in the right-hand side does not depend on z , call it c_N . The third summand converges to $-(z - i)d(R)$. Thus, we have the convergence $S(z) - c_N \rightarrow S_*(z)$ to a holomorphic function, uniformly in z in compact subsets in \mathbb{H} . One can check that the derivative of S_* is (6.9).

Observe that the length of the part of the contour \mathcal{C}_w^ϵ close to the real line is bounded. From the convergence of $S(z) - c_N$ it follows that away from the real line the contour \mathcal{C}_w^ϵ approximates the corresponding contour for $S_*(z)$, and the latter has finite length. Thus, we get a contradiction. \square

Remark 6.9. The argument in the beginning of the proof of Lemma 6.8 shows that the additional hypotheses of Theorem 2.10, namely, convergence of measures (6.8) and convergence $d(R) = \lim_{N \rightarrow +\infty} d_N(R)$, are not too restrictive.

Proof of Theorem 2.10. By Hurwitz's theorem, the critical points $z_c(N) \in \mathbb{H}$ of S' converge, as $N \rightarrow +\infty$, to a critical point $z_* \in \mathbb{H}$, which belongs to the compact set \mathcal{Z} from Theorem 2.7. Applying the latter, we get the desired convergence. Equation (2.13) is simply $S'_*(z) = 0$ under a change of variables $z = u/(1 - u)$, which maps the upper half plane onto itself. \square

As Proposition 2.18 is a particular case of Theorem 2.10, let us give its proof here:

Proof of Proposition 2.18. Under the hypotheses of Proposition 2.18, Assumptions 1 and 2 clearly hold, and, moreover, $d_N(R)$ is close to zero for $R > 3$. Therefore, the function $S'_*(z)$ (6.9) looks as

$$\begin{aligned} S'_*(z) &= \frac{1}{2} \int_{-3h}^0 \frac{dv}{z - v} + \frac{1}{3} \int_0^{3h} \frac{dv}{z - v} + \frac{1}{2} \text{p.v.} \int_{|v| > 3h} \frac{dv}{z - v} \\ &\quad + \log(z + 1) - \log z + i\pi - \log(\beta^{-1} - 1) \\ &= \frac{i\pi}{2} - \frac{1}{6} \log z + \frac{1}{6} \log(z - 3h) + \log(z + 1) - \log z - \log(\beta^{-1} - 1). \end{aligned}$$

After the substitution $z = u/(1 - u)$ this leads to the equation (2.20) for the complex slope. \square

Proof of Theorem 2.11. Fix an arbitrary $\varepsilon > 0$. Choose and fix $\delta > 0$ so small that $A_{R,\delta} < \varepsilon/3$ and that

$$\left| \int_{|v| > \delta} \frac{\mu_{glob}(dv)}{v} - \text{p.v.} \int_{-\infty}^{\infty} \frac{\mu_{glob}(dv)}{v} \right| < \frac{\varepsilon}{3}.$$

This approximation is possible because μ_{glob} is a probability measure, so $1/v$ is integrable at infinity, and thus the only singularity is at zero. Next, let N be so large that

$$\left| \sum_{i: |a_i(N)| > \delta N} \frac{1}{a_i(N)} - \int_{|u| > \delta} \frac{\mu_{glob}(dv)}{v} \right| < \frac{\varepsilon}{3}.$$

This is possible because the sum above is the same as the integral of $\mathbf{1}_{|v|>\delta}/v$ with respect to the atomic measure $\mu_{glob}^N = \frac{1}{N} \sum_{i=1}^N \delta_{a_i(N)/N}$ converging vaguely to μ_{glob} . Here the vague convergence implies the convergence on the function $\mathbf{1}_{|v|>\delta}/v$ because we are dealing with probability measures and so can cut away the tails at infinity. These estimates imply that

$$\mathbf{d}_N(\mathbf{R}) = \sum_{i: \mathbf{R}T \leq |a_i(N)| \leq \delta N} \frac{1}{a_i(N)} + \int_{|v|>\delta} \frac{\mu_{glob}^N(dv)}{v}$$

is close to p.v. $\int_{-\infty}^{\infty} v^{-1} \mu_{glob}(dv)$ within ε , and so an application of Theorem 2.10 gives the result. \square

Proof of Theorem 2.12. When μ_{loc} is a multiple of the Lebesgue measure, the integral in (6.9) can be explicitly computed:

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{1}{z-v} + \frac{\mathbf{1}_{|v|>\mathbf{R}}}{v} \right) dv &= \int_{-\mathbf{R}}^{\mathbf{R}} \frac{dv}{z-v} + \lim_{M \rightarrow +\infty} \int_{\mathbf{R} < |v| < M} \left(\frac{1}{z-v} + \frac{1}{v} \right) dv \\ &= \log(z+\mathbf{R}) - \log(z-\mathbf{R}) + \lim_{M \rightarrow +\infty} \left(\log(z+M) - \log(z-M) + \log(z-\mathbf{R}) - \log(z+\mathbf{R}) \right) \\ &= \lim_{M \rightarrow +\infty} \left(\log(z+M) - \log(z-M) \right), \end{aligned}$$

where the branches of all the logarithms above are standard. The last limit is equal to $-\mathbf{i}\pi$ because $\arg(z+M) \rightarrow 0$ while $\arg(z-M) \rightarrow \pi$. This immediately leads to the desired formula for \mathbf{u}_* in Theorem 2.12. \square

Remark 6.10. When μ_{loc} is a multiple of the Lebesgue measure, the above computation shows that the integral in (6.9) is independent of \mathbf{R} , and hence $\mathbf{d}(\mathbf{R})$ is, too. This agrees with the fact that the difference $\mathbf{d}_N(\mathbf{R}) - \mathbf{d}_N(\mathbf{R}')$ (where, say, $\mathbf{R} > \mathbf{R}'$) is equal to the integral of $\mathcal{M}_T[\mathbf{a}](dv)/v$ over $\mathbf{R}' < |v| < \mathbf{R}$, and thus vanishes as $N \rightarrow +\infty$.

7. APPLICATIONS: PROOFS OF THEOREMS 2.13 AND 2.15 AND PROPOSITION 2.16

7.1. Discretization of a continuous profile: proof of Theorem 2.13. Let us show that the initial data (2.18) defined using a twice continuously differentiable function f satisfies Assumptions 1 and 2, as well as additional hypotheses of Theorem 2.10.

Since $1 < f'(x) < +\infty$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$, Assumption 1 holds with

$$\rho_{\bullet} = \frac{1}{2} \inf_{x \in [-\frac{1}{2}, \frac{1}{2}]} \frac{1}{f'(x)}, \quad \rho^{\bullet} = \frac{1}{2} + \frac{1}{2} \sup_{x \in [-\frac{1}{2}, \frac{1}{2}]} \frac{1}{f'(x)}$$

on scales, say, $\mathbf{D}(N) = \lfloor N^{\eta/2} \rfloor$ and $\mathbf{Q}(N) = \lfloor N^{(1+\eta)/2} \rfloor$.

Next, the local and global measures as in (2.14) exist, μ_{loc} is the Lebesgue measure on \mathbb{R} times $q = 1/f'(\chi)$ (recall that $f(\chi) = 0$), and μ_{glob} has the density

$$\mu_{glob}(dv) = \frac{dv}{f'(f^{-1}(v))}.$$

Because $f' > 1$, the principal value integral

$$\text{p.v.} \int_{-\infty}^{+\infty} \frac{dv}{v f'(f^{-1}(v))} = \text{p.v.} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{f(x)} \quad (7.1)$$

also exists.

Let us now consider the quantities $\mathbf{d}_N(\mathbf{R})$ (2.12). Replace the condition $|a_i| \geq RT$ on i in (2.12) by $|i - N\chi| > RT/f'(\chi)$. The difference between the two sums can be estimated by a part of the harmonic series between RT and $RT + CT^2/N$, which is negligible. Thus,

$$\lim_{N \rightarrow +\infty} \mathbf{d}_N(Rf'(\chi)) = \lim_{N \rightarrow +\infty} \sum_{i: |i - N\chi| > RT} \frac{1}{[Nf(i/N)]} = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i: |i/N - \chi| > RT/N} \frac{1}{f(i/N)}. \quad (7.2)$$

Taylor expand $f(i/N) = (i/N - \chi)f'(\chi) + (i/N - \chi)^2 r$, where $|r|$ is uniformly bounded (here we use that f is twice continuously differentiable). Observe that the sum of

$$\frac{1}{Nf(i/N)} - \frac{1}{(i - N\chi)f'(\chi)} = -\frac{r}{N} \cdot \frac{1}{f'(\chi)^2 + rf'(\chi)(i/N - \chi)}$$

over i such that $RT/N < i/N - \chi < \delta$ (i.e., the one-sided sum) is bounded for sufficiently small $\delta > 0$, and goes to zero as $\delta \rightarrow 0$, and similarly for $-\delta < i/N - \chi < -RT/N$. The sum of $1/((i - N\chi)f'(\chi))$ over $|i - N\chi| > RT$ (i.e., the symmetric sum) is negligible for large N . Thus, the last sum in (7.2) over $|i/N - \chi| > RT/N$ is close to the same sum over $|i/N - \chi| > \delta$ for small δ , and the latter approximates the principal value integral (7.1). This implies Assumption 2 and the property that the $\mathbf{d}_N(\mathbf{R})$'s converge as $N \rightarrow +\infty$. This completes the proof of Theorem 2.13.

7.2. Random initial configuration. Let us now consider the noncolliding Bernoulli random walk started from a random initial configuration. We assume that this random configuration belongs to $\{-M, -M + 1, \dots, M\}$, where $M \rightarrow +\infty$ is our main large parameter. Denote by

$$\mathbb{W}(M) = \bigcup_{k=0}^{2M+1} \{\vec{x} \in \mathbb{W}^k : -M \leq x_1 < \dots < x_k \leq M\} \quad (7.3)$$

the space of possible initial configurations (cf. (1.1)). The law of the initial configuration will be denoted by \mathbf{P}^M , and the configuration itself by $\mathfrak{A}(\mathbf{N})$ (here \mathbf{N} can be random). Let $\vec{\mathbf{X}}(t)$ stand for the noncolliding Bernoulli random walk started from $\mathfrak{A}(\mathbf{N})$.

When does $\vec{\mathbf{X}}(t)$ satisfy an annealed¹⁴ bulk limit theorem similar to the ones formulated in Section 2.3? Informally speaking, this happens when the walk started from a fixed “typical” configuration (with respect to \mathbf{P}^M) satisfies a bulk limit theorem with a constant complex slope u_* (i.e., independent of the randomness coming from the random initial data). Let us formalize this understanding, cf. [DJM15], [Gor16] for recent annealed limit theorems for uniformly random tilings.

¹⁴That is, with respect to the combined randomness coming from the initial configuration and from the random walk itself.

Proposition 7.1. *Choose and fix a time scale $T = T(M) \ll M$, $T(M) \rightarrow +\infty$. Suppose that there exist subsets $\mathbb{W}_{reg}(M) \subset \mathbb{W}(M)$, $M = 1, 2, \dots$, such that*

1. $\lim_{M \rightarrow +\infty} \mathbf{P}^M(\mathbb{W}_{reg}(M)) = 1$;
2. *For any fixed sequence of (nonrandom) initial configurations $\mathfrak{A}(N_M) \in \mathbb{W}_{reg}(M)$ the bulk limit theorem near $x = 0$ (i.e., the conclusion of Theorem 2.10) holds for a complex slope \mathbf{u}_* independent of this sequence $\mathfrak{A}(N_M)$.*

Then as $M \rightarrow +\infty$ the point process describing $\{\vec{\mathbf{X}}(T(M) + t)\}_t$ near $x = 0$ converges in distribution to the extended sine process of the complex slope \mathbf{u}_ .*

In particular, hypotheses of Proposition 7.1 imply that the random number \mathbf{N} of particles in the initial configuration goes to infinity in probability (with respect to the \mathbf{P}^M 's).

Proof of Proposition 7.1. Fix an event of the form

$$F = \left\{ \text{the configuration } \vec{\mathbf{X}}(T + t_i) \text{ on } \mathbb{Z} \text{ contains the point } y_i \text{ for all } i = 1, \dots, k \right\},$$

where $k = 1, 2, \dots$, and $t_i, y_i \in \mathbb{Z}$ (cf. (2.1)). Such events generate the σ -algebra describing the configuration $\{\vec{\mathbf{X}}(T + t)\}_t$ near $x = 0$.

Let $\mathbb{P}_{\mathfrak{A}(N)}$ stand for the law of the noncolliding Bernoulli random walk started from the initial configuration $\mathfrak{A}(N)$. We need to show that

$$\lim_{M \rightarrow +\infty} \mathbf{E}_{\mathbf{P}^M} [\mathbb{P}_{\mathfrak{A}(\mathbf{N})}(F)] = \det [K_{\mathbf{u}_*}(t_\alpha, y_\alpha; t_\beta, y_\beta)]_{\alpha, \beta=1}^k, \quad (7.4)$$

where $K_{\mathbf{u}_*}$ is the extended sine kernel (2.3), and $\mathbf{E}_{\mathbf{P}^M}$ denotes the expectation with respect to \mathbf{P}^M . We have

$$\mathbf{E}_{\mathbf{P}^M} [\mathbb{P}_{\mathfrak{A}(\mathbf{N})}(F)] = \mathbf{E}_{\mathbf{P}^M} [\mathbb{P}_{\mathfrak{A}(\mathbf{N})}(F) \mathbf{1}_{\mathbb{W}_{reg}(M)}] + \mathbf{E}_{\mathbf{P}^M} [\mathbb{P}_{\mathfrak{A}(\mathbf{N})}(F) \mathbf{1}_{\mathbb{W}(M) \setminus \mathbb{W}_{reg}(M)}].$$

The second summand goes to zero by hypothesis 1, and the first summand can be estimated as

$$\mathbb{P}_{\mathfrak{A}_{\min}(N_{\min})}(F) \leq \mathbf{E}_{\mathbf{P}^M} [\mathbb{P}_{\mathfrak{A}(\mathbf{N})}(F) \mathbf{1}_{\mathbb{W}_{reg}(M)}] \leq \mathbb{P}_{\mathfrak{A}_{\max}(N_{\max})}(F), \quad (7.5)$$

where $\mathfrak{A}_{\max}(N_{\max})$ is the configuration which maximizes $\mathbb{P}_{\mathfrak{A}}(F)$ over all $\mathfrak{A} \in \mathbb{W}_{reg}(M)$ (it exists because this is a finite set), and similarly for $\mathfrak{A}_{\min}(N_{\min})$. Because both minimizing and maximizing configurations belong to $\mathbb{W}_{reg}(M)$, both bounds in (7.5) converge to the right-hand side of (7.4) by hypothesis 2, and so the desired convergence holds. \square

7.3. Bernoulli initial data: proof of Theorem 2.15. Let the parameters p and α be as in Theorem 2.15, the time scale be $T(M) = \lfloor M^\eta \rfloor$, and the initial particle configuration on $\{-\lfloor M(1 - \alpha) \rfloor, -\lfloor M(1 - \alpha) \rfloor + 1, \dots, \lfloor M\alpha \rfloor - 1, \lfloor M\alpha \rfloor\}$ be obtained by putting a particle at each location with probability p independently of all others.

We will construct a subset $\mathbb{W}_{reg}(M) \subset \mathbb{W}(M)$ (7.3) satisfying Proposition 7.1 by imposing two conditions (of asymptotic \mathbf{P}^M -probability 1) on the configuration. First, fix $0 < \delta < \min(1 - \eta, \eta/7)$, and take scales $\mathbf{D}(M) = \lfloor M^{\eta - \delta} \rfloor$, $\mathbf{Q}(M) = \lfloor M^{\eta + \delta} \rfloor$. Denote

$$\mathbb{W}_{reg,1}(M) = \left\{ \vec{x} \in \mathbb{W}(M) \text{ such that in every segment of length } \mathbf{D}(M) \right. \\ \left. \text{inside } [-\mathbf{Q}(M), \mathbf{Q}(M)] \text{ the number of points in the con-} \right. \\ \left. \text{figuration } \vec{x} \text{ is between } p\mathbf{D} - \mathbf{D}^{2/3} \text{ and } p\mathbf{D} + \mathbf{D}^{2/3} \right\}.$$

Since the expected number of points in one of the segments of length D is pD and the variance is of order D , by the Chebyshev inequality the probability that in one of such segments the (random) number of points is not between $pD - D^{2/3}$ and $pD + D^{2/3}$ can be bounded from above by a constant times $D^{-1/3}$. The number of segments of length D inside $[-Q, Q]$ is of order Q/D , and so

$$1 - \mathbf{P}^M(\mathbb{W}_{reg,1}(M)) \leq \text{const} \cdot D^{-\frac{1}{3}} \frac{Q}{D} = \text{const} \cdot M^{-\frac{4}{3}(\eta-\delta)+\eta+\delta} = \text{const} \cdot M^{\frac{1}{3}(7\delta-\eta)} \rightarrow 0.$$

Clearly, configurations in $\mathbb{W}_{reg,1}(M)$ satisfy Assumption 1. Moreover, for these configurations the local density of particles at 0 vaguely converges as $M \rightarrow +\infty$ to p times the Lebesgue measure on \mathbb{R} .

Second, recall $\mathbf{d}_M(\mathbf{R})$ defined by (2.12) as a sum of $1/a_i$ over $|a_i| \geq RT(M)$, and interpret it as a sum of independent random variables δ_i/i over all $i \in \{-\lfloor M(1-\alpha) \rfloor, \dots, \lfloor M\alpha \rfloor\}$, where δ_i is the indicator of the event that there is a point of the configuration at the location i . We have

$$\mathbf{E}_{\mathbf{P}^M}(\mathbf{d}_M(\mathbf{R})) = p \sum_{\substack{j: |j| \geq RT(M) \\ -\lfloor M(1-\alpha) \rfloor \leq j \leq \lfloor M\alpha \rfloor}} \frac{1}{j}, \quad \text{Var}_{\mathbf{P}^M}(\mathbf{d}_M(\mathbf{R})) = p(1-p) \sum_{\substack{j: |j| \geq RT(M) \\ -\lfloor M(1-\alpha) \rfloor \leq j \leq \lfloor M\alpha \rfloor}} \frac{1}{j^2}.$$

We see that the expectation approximates an integral

$$\mathbf{E}_{\mathbf{P}^M}(\mathbf{d}_M(\mathbf{R})) = \text{p.v.} \int_{-(1-\alpha)}^{\alpha} \frac{p}{v} dv + O(M^{-1}) = p \log \left(\frac{\alpha}{1-\alpha} \right) + O(M^{-1}),$$

and $\text{Var}_{\mathbf{P}^M}(\mathbf{d}_M(\mathbf{R})) = O(T(M)^{-1})$, so the random variable $\mathbf{d}_M(\mathbf{R})$ converges as $M \rightarrow +\infty$ to the constant $\mathbf{d} = p \log(\frac{\alpha}{1-\alpha})$. Thus, if we define for some fixed $R > 0$:

$$\mathbb{W}_{reg,2}(M) = \left\{ \vec{x} \in \mathbb{W}(M) : \left| p \log \left(\frac{\alpha}{1-\alpha} \right) - \sum_{i: |x_i| \geq RT(M)} \frac{1}{x_i} \right| < M^{-\eta/3} \right\},$$

then by the Chebyshev inequality we have $\mathbf{P}^M(\mathbb{W}_{reg,2}(M)) \rightarrow 1$. Configurations from the sets $\mathbb{W}_{reg,2}(M)$ satisfy Assumption 2, and have $\mathbf{d}_M(\mathbf{R}) \rightarrow \mathbf{d}$.

Defining $\mathbb{W}_{reg}(M) = \mathbb{W}_{reg,1}(M) \cap \mathbb{W}_{reg,2}(M)$, we see that the \mathbf{P}^M -probabilities of these sets go to 1, while any sequence of configurations from $\mathbb{W}_{reg}(M)$ satisfies the hypotheses of Theorem 2.10 and hence the bulk limit theorem near $x = 0$. Thus, applying Proposition 7.1 we see that Theorem 2.15 is established.

7.4. Sine process initial data: proof of Proposition 2.16. Let the parameters ϕ and α be as in Proposition 2.16, the time scale be $T(M) = \lfloor M^\eta \rfloor$, and the initial particle configuration be obtained by restricting the configuration of the discrete sine process of density ϕ/π to $\{-\lfloor M(1-\alpha) \rfloor, -\lfloor M(1-\alpha) \rfloor + 1, \dots, \lfloor M\alpha \rfloor - 1, \lfloor M\alpha \rfloor\}$. We will use the same scales $D(M), Q(M)$ and sets $\mathbb{W}_{reg,1}(M), \mathbb{W}_{reg,2}(M)$ as in the Bernoulli case in Section 7.3 with p replaced by ϕ/π . This ensures that the second hypothesis of Proposition 7.1 holds for $\mathbb{W}_{reg}(M) = \mathbb{W}_{reg,1}(M) \cap \mathbb{W}_{reg,2}(M)$. To establish Proposition 2.16 it remains to show that

$$\lim_{M \rightarrow +\infty} \mathbf{P}^M(\mathbb{W}_{reg,1}(M)) = \lim_{M \rightarrow +\infty} \mathbf{P}^M(\mathbb{W}_{reg,2}(M)) = 1,$$

where now \mathbf{P}^M stands for the law of the initial configuration under the restriction of the discrete sine process. For $\mathbb{W}_{reg,1}(M)$, observe that the variance under \mathbf{P}^M of the number of points in a segment of length D (say, $\{1, \dots, D\}$, since the sine process is translation invariant) can be estimated as

$$\begin{aligned} \text{Var}_{\mathbf{P}^M} \left(\sum_{i=1}^D \delta_i \right) &= \frac{\phi}{\pi} D - \left(\frac{\phi}{\pi} D \right)^2 + 2 \sum_{1 \leq i < j \leq D} \mathbf{E}_{\mathbf{P}^M}(\delta_i \delta_j) \\ &= D \frac{\phi}{\pi} \left(1 - \frac{\phi}{\pi} \right) - 2 \sum_{1 \leq i < j \leq D} \left(\frac{\sin(\phi(i-j))}{\pi(i-j)} \right)^2 \leq D \frac{\phi}{\pi} \left(1 - \frac{\phi}{\pi} \right), \end{aligned} \quad (7.6)$$

where δ_i is as in Section 7.3. This variance does not exceed the one in the Bernoulli case, and so using the argument from Section 7.3 we conclude that $\mathbf{P}^M(\mathbb{W}_{reg,1}(M)) \rightarrow 1$.

Remark 7.2. In fact, the variance in the left-hand side of (7.6) grows as $O(\log D)$, see [BF14, Lemma 4.6], [Buf12, Section 4.2], but we do not need this for our proof.

For $\mathbb{W}_{reg,2}(M)$ consider the random variable $\mathbf{d}_M(\mathbf{R})$ (2.12). Arguing as in Section 7.3, we see that its expectation converges to $\frac{\phi}{\pi} \log(\frac{\alpha}{1-\alpha})$. Let us estimate its variance. Observe that for any subset $B \subset \mathbb{Z} \setminus \{0\}$ one has

$$\text{Var}_{\mathbf{P}^M} \left(\sum_{i \in B} \frac{\delta_i}{i} \right) = \frac{\phi}{\pi} \left(1 - \frac{\phi}{\pi} \right) \sum_{i \in B} \frac{1}{i^2} - 2 \sum_{i, j \in B: i < j} \frac{1}{ij} \left(\frac{\sin(\phi(i-j))}{\pi(i-j)} \right)^2. \quad (7.7)$$

Apply this with

$$B = \{-\lfloor M(1-\alpha) \rfloor, \dots, -RT-1, -RT, \dots, RT, RT+1, \dots, \lfloor M\alpha \rfloor - 1, \lfloor M\alpha \rfloor\}$$

for some fixed $R \in \mathbb{Z}_{\geq 1}$. We see that as $M \rightarrow +\infty$, the first sum in (7.7) decays as $O(T^{-1})$. Throwing away the pairs (i, j) of the same sign from the second sum, we can bound the second sum in (7.7) by a constant times

$$\sum_{k, j \geq RT} \frac{1}{kj(k+j)^2} \leq \sum_{j \geq RT} \left(\sum_{k \geq 1} \frac{1}{kj(k+j)^2} \right) \leq \sum_{j \geq RT} \frac{C_1 + C_2 \log j}{j^2} = O(T^{-1+\epsilon}) \quad (7.8)$$

for some $C_{1,2} > 0$ and an arbitrary small $\epsilon > 0$. Thus, the variance of $\mathbf{d}_M(\mathbf{R})$ decays as $O(T^{-1+\epsilon})$, and so by the Chebyshev inequality we have $\mathbf{P}^M(\mathbb{W}_{reg,2}(M)) \rightarrow 1$. Applying Proposition 7.1, we see that Proposition 2.16 holds.

Remark 7.3. One can say that the constant $\mathbf{d} = \frac{\phi}{\pi} \log(\frac{\alpha}{1-\alpha})$ in Proposition 2.16 corresponds via (2.15) to the global probability measure μ_{glob} which is the uniform measure on the segment $[-(1-\alpha)\frac{\pi}{\phi}, \alpha\frac{\pi}{\phi}]$. Indeed, this μ_{glob} is a limit as in (2.14) of random atomic measures corresponding to the sine process initial data, where as N one should take the random number of particles \mathbf{N} (it is concentrated around $\frac{\phi}{\pi}M$). Similarly, the constant $p \log(\frac{\alpha}{1-\alpha})$ in Theorem 2.15 corresponds to μ_{glob} being the uniform measure on $[-(1-\alpha)p^{-1}, \alpha p^{-1}]$.

Remark 7.4. The proof of Proposition 2.16 carries over from the Bernoulli case modulo two estimates of the variance (7.6) and (7.7)–(7.8), which are rather straightforward for the sine process. Thus, the bulk limit theorem should hold for rather general random initial data, but we will not formulate any other results in this direction.

APPENDIX A. DETERMINANTAL KERNELS FOR OTHER NONCOLLIDING PROCESSES

A.1. Noncolliding Poisson random walk. Taking the limit as $\beta \rightarrow 0$ and scaling to the continuous time as $t = \lfloor \beta^{-1} \tau \rfloor$, $\tau \in \mathbb{R}_{\geq 0}$, turns the noncolliding Bernoulli random walk into the noncolliding Poisson random walk — the continuous time dynamics of N independent speed 1 Poisson particles conditioned to never collide [KOR02]. This Markov chain $\vec{X}(\tau)$ on \mathbb{W}^N has jump rates (cf. (1.2))

$$\mathbb{P}(\vec{X}(\tau + d\tau) = \vec{x}' \mid \vec{X}(\tau) = \vec{x}) = \begin{cases} \frac{V(\vec{x}')}{V(\vec{x})} d\tau + O(d\tau^2), & \begin{aligned} &x'_i = x_i + 1 \text{ for some} \\ &i, \text{ and } x'_j = x_j \text{ for} \\ &j \neq i; \end{aligned} \\ 1 - Nd\tau + O(d\tau^2), & \vec{x}' = \vec{x}; \\ 0, & \text{otherwise,} \end{cases}$$

where $\vec{x}, \vec{x}' \in \mathbb{W}^N$ are arbitrary.¹⁵ The noncolliding Poisson random walk is also sometimes referred to as the Charlier process [BF14] due to the fact that if it starts from the densely packed initial configuration $(0, 1, \dots, N-1)$, then its (fixed time) distribution is the Charlier orthogonal polynomial ensemble (cf. Remark 2.2).

Theorem A.1. *The noncolliding Poisson random walk $\vec{X}(\tau)$ started from an arbitrary initial configuration $\vec{a} \in \mathbb{W}^N$ is determinantal in the sense of (2.1), with the kernel*

$$K_{\vec{a}}^{\text{Poisson}}(\tau_1, x_1; \tau_2, x_2) = -\mathbf{1}_{x_1 \geq x_2} \mathbf{1}_{\tau_1 > \tau_2} \frac{(\tau_1 - \tau_2)^{x_1 - x_2}}{(x_1 - x_2)!} \\ - \frac{1}{(2\pi i)^2} \int_{x_2 - \frac{1}{2} - i\infty}^{x_2 - \frac{1}{2} + i\infty} dz \oint_{\text{all } w \text{ poles}} dw \frac{1}{w - z} \frac{\Gamma(x_2 - z)}{\Gamma(x_1 - w + 1)} \tau_1^{x_1 - w} \tau_2^{z - x_2} \prod_{r=1}^N \frac{z - a_r}{w - a_r}, \quad (\text{A.1})$$

where $x_{1,2} \in \mathbb{Z}$, $\tau_{1,2} > 0$, the z integration contour is a vertical line $\text{Re } z = x_2 - \frac{1}{2}$ traversed upwards, and the w contour is a positively oriented circle or a union of two circles encircling all the w poles $\{\dots, x_1 - 1, x_1\} \cap \{a_1, \dots, a_N\}$ of the integrand except $w = z$.

Proof. We will obtain $K_{\vec{a}}^{\text{Poisson}}$ from $K_{\vec{a};\beta}^{\text{Bernoulli}}$ (2.2) via the $\beta \rightarrow 0$ limit described above. Employing Remark 3.7, write $K_{\vec{a};\beta}^{\text{Bernoulli}}$ as

$$K_{\vec{a};\beta}^{\text{Bernoulli}}(t_1, x_1; t_2, x_2) = \mathbf{1}_{x_1 \geq x_2} \mathbf{1}_{t_1 > t_2} (-1)^{x_1 - x_2 + 1} \binom{t_1 - t_2}{x_1 - x_2} + \frac{t_1!}{(t_2 - 1)!} \frac{1}{(2\pi i)^2}$$

¹⁵In particular, this implies $\sum_{i=1}^N V(\vec{x} + e_i) = NV(\vec{x})$, where e_i is the i -th basis vector $(0, \dots, 0, 1, 0, \dots, 0)$.

$$\times \int_{x_2 - \frac{1}{2} - i\infty}^{x_2 - \frac{1}{2} + i\infty} dz \oint_{\text{all } w \text{ poles}} dw \frac{(z - x_2 + 1)_{t_2-1}}{(w - x_1)_{t_1+1}} \frac{1}{w - z} \frac{\sin(\pi w)}{\sin(\pi z)} \left(\frac{1 - \beta}{\beta} \right)^{w-z} \prod_{r=1}^N \frac{z - a_r}{w - a_r}. \quad (\text{A.2})$$

Here the w contour (a circle or a union of two circles) can be taken to encircle the points $\{a_1, a_2, \dots, a_N\}$, which contain all the w poles except $w = z$. (Indeed, for $w = a_i$ to be a pole, it must additionally satisfy $(a_i - x_1)_{t_1+1} = 0$ to not cancel with the zero coming from $\sin(\pi w)$.) Therefore, the integration contours do not depend on $t_{1,2}$, and we can take the Poisson rescaling of (A.2), that is, $\beta \rightarrow 0$ and $t_{1,2} = \lfloor \beta^{-1} \tau_{1,2} \rfloor$ with $\tau_1 \geq 0$, $\tau_2 > 0$.

The t -dependent part of the first summand in (A.2) scales as

$$\begin{aligned} \mathbf{1}_{\tau_1 > \tau_2} \binom{\lfloor \beta^{-1} \tau_1 \rfloor - \lfloor \beta^{-1} \tau_2 \rfloor}{x_1 - x_2} &= \mathbf{1}_{\tau_1 > \tau_2} \frac{\Gamma(\lfloor \beta^{-1} \tau_1 \rfloor - \lfloor \beta^{-1} \tau_2 \rfloor + 1)}{\Gamma(\lfloor \beta^{-1} \tau_1 \rfloor - \lfloor \beta^{-1} \tau_2 \rfloor - x_1 + x_2 + 1)(x_1 - x_2)!} \\ &\sim \mathbf{1}_{\tau_1 > \tau_2} \frac{(\tau_1 - \tau_2)^{x_1 - x_2}}{(x_1 - x_2)!} \beta^{-(x_1 - x_2)}, \end{aligned}$$

where we used (3.7). Similarly, for the part of the integrand depending on β and $t_{1,2}$ we have

$$\begin{aligned} \frac{t_1!}{(t_2 - 1)!} \frac{(z - x_2 + 1)_{t_2-1}}{(w - x_1)_{t_1+1}} \left(\frac{1 - \beta}{\beta} \right)^{w-z} \frac{\sin(\pi w)}{\sin(\pi z)} \\ = \frac{\Gamma(t_1 + 1)}{\Gamma(w - x_1 + t_1 + 1)} \frac{\Gamma(z - x_2 + t_2)}{\Gamma(t_2)} \frac{\Gamma(w - x_1)}{\Gamma(z - x_2 + 1)} \left(\frac{1 - \beta}{\beta} \right)^{w-z} \frac{\sin(\pi w)}{\sin(\pi z)} \\ \sim \frac{\Gamma(x_2 - z)}{\Gamma(x_1 - w + 1)} (-1)^{x_1 - x_2 + 1} \tau_1^{x_1 - w} \tau_2^{z - x_2} \beta^{-(x_1 - x_2)}. \end{aligned}$$

Note that $(1 - \beta)^{w-z} \rightarrow 1$ as $\beta \rightarrow 0$, and in the last step we used (3.12). This implies

$$\lim_{\beta \rightarrow 0} \left((-\beta)^{x_1 - x_2} K_{\vec{a}; \beta}^{\text{Bernoulli}}(\lfloor \beta^{-1} \tau_1 \rfloor, x_1; \lfloor \beta^{-1} \tau_1 \rfloor, x_2) \right) = K_{\vec{a}}^{\text{Poisson}}(\tau_1, x_1; \tau_2, x_2),$$

where $K_{\vec{a}}^{\text{Poisson}}$ is given by (A.1). Because the multiplication by $(-\beta)^{x_1 - x_2}$ does not change the correlation functions (cf. footnote⁵), this completes the proof. \square

The correlation kernel of Theorem A.1 appears to be new. By analogy with the results in Section 2.3, we believe that the local statistics of the noncolliding Poisson random walk are universally described by an extension of the discrete sine kernel with the continuous time parameter. This extension first appeared in [BO06], see also [Bor07], [BS10] for a general discussion of extensions of the discrete sine kernel. We will not pursue this in the present paper.

A.2. Dyson Brownian Motion. A diffusion scaling brings the kernel $K_{\vec{a}; \beta}^{\text{Bernoulli}}$ (2.2) to the kernel of the Dyson Brownian Motion. We will use the following scaling (where $M \rightarrow +\infty$):

$$t_{1,2} = \lfloor M \tau_{1,2} \rfloor, \quad x_{1,2} = \lfloor \beta M \tau_{1,2} + \xi_{1,2} \sqrt{\beta(1 - \beta)} \sqrt{M} \rfloor, \quad a_i = \lfloor \alpha_i \sqrt{\beta(1 - \beta)} \sqrt{M} \rfloor,$$

where $(\alpha_1 \leq \dots \leq \alpha_N) \in \mathbb{R}^N$ are the rescaled starting points (by agreement, when some of the α_i 's coincide, the corresponding a_i 's differ by 1, so that the discrete noncolliding Bernoulli random walk is well-defined).

Theorem A.2. *Under the above scaling and up to a gauge transformation as in footnote⁵, the kernel $(M\beta(1-\beta))^{\frac{1}{2}} K_{\vec{a};\beta}^{\text{Bernoulli}}$ converges to the following kernel:*

$$K_{\vec{a};\beta}^{\text{DBM}}(\tau_1, \xi_1; \tau_2, \xi_2) = -\frac{\mathbf{1}_{\tau_1 > \tau_2}}{\sqrt{2\pi\Delta\tau}} \exp\left\{-\frac{(\Delta\xi)^2}{2\Delta\tau}\right\} \\ - \frac{1}{(2\pi\mathbf{i})^2 \sqrt{\tau_1\tau_2}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} dz \oint_{\text{all } w \text{ poles}} dw \frac{1}{w-z} \exp\left\{\frac{\tau_1(z-\xi_2)^2 - \tau_2(w-\xi_1)^2}{2\tau_1\tau_2}\right\} \prod_{r=1}^N \frac{z-\alpha_r}{w-\alpha_r}, \quad (\text{A.3})$$

where $\xi_{1,2} \in \mathbb{R}$, $\tau_{1,2} > 0$, and we use the notation $\Delta\xi = \xi_1 - \xi_2$, $\Delta\tau = \tau_1 - \tau_2$. The z contour is a vertical line which lies to the left of all the α_r 's (i.e., $c < \alpha_1$), and the w contour is a positively oriented circle encircling all the α_r 's.

The multiplication by $(M\beta(1-\beta))^{\frac{1}{2}}$ corresponds to the rescaling of the space from discrete to continuous (the correlation kernel should be viewed as a kernel of an integral operator). Note also that the kernel (A.3) can similarly be obtained as a diffusion limit of the Poisson kernel (A.1), but we will not perform this computation.

Proof of Theorem A.2. Let us denote $\sigma = \sqrt{\beta(1-\beta)}$ to shorten the notation. Changing the variables as $z = \tilde{z}\sigma\sqrt{M}$, $w = \tilde{w}\sigma\sqrt{M}$, and renaming back to z, w , we can rewrite (2.2) as

$$K_{\vec{a};\beta}^{\text{Bernoulli}}(t_1, x_1; t_2, x_2) = \mathbf{1}_{x_1 \geq x_2} \mathbf{1}_{t_1 > t_2} (-1)^{x_1 - x_2 + 1} \binom{t_1 - t_2}{x_1 - x_2} \\ + \sigma\sqrt{M} \frac{t_1!}{(t_2 - 1)!} \frac{1}{(2\pi\mathbf{i})^2} \int_{(x_2 - t_2 + \frac{1}{2})/(\sigma\sqrt{M}) - \mathbf{i}\infty}^{(x_2 - t_2 + \frac{1}{2})/(\sigma\sqrt{M}) + \mathbf{i}\infty} dz \oint_{\text{all } w \text{ poles}} dw \frac{(z\sigma\sqrt{M} - x_2 + 1)_{t_2-1}}{(w\sigma\sqrt{M} - x_1)_{t_1+1}} \\ \times \frac{1}{w-z} \frac{\sin(\pi w\sigma\sqrt{M})}{\sin(\pi z\sigma\sqrt{M})} \left(\frac{1-\beta}{\beta}\right)^{(w-z)\sigma\sqrt{M}} \prod_{r=1}^N \frac{z - \lfloor \alpha_r \sigma\sqrt{M} \rfloor / (\sigma\sqrt{M})}{w - \lfloor \alpha_r \sigma\sqrt{M} \rfloor / (\sigma\sqrt{M})}. \quad (\text{A.4})$$

The w contour encircles all the w poles of the integrand except $w = z$, which are close to the points $(\alpha_1 \leq \dots \leq \alpha_N)$. For large M the z contour will get shifted further to the left; the computation below shows that there will be no poles crossed while doing this.

Denote

$$c_{1,2} = t_{1,2} - M\tau_{1,2}, \quad d_{1,2} = x_{1,2} - \beta M\tau_{1,2} - \xi_{1,2}\sigma\sqrt{M},$$

these are numbers between -1 and 0 . Let us first consider asymptotics of the non-integral summand. For large M , the two indicators reduce simply to $\mathbf{1}_{\tau_1 > \tau_2}$, and the binomial coefficient has the asymptotics

$$\begin{aligned} \binom{t_1 - t_2}{x_1 - x_2} &= \frac{\Gamma(t_1 - t_2 + 1)}{\Gamma(x_1 - x_2 + 1)\Gamma(t_1 - t_2 - (x_1 - x_2) + 1)} = \frac{1/\sqrt{M}}{\sqrt{2\pi\sigma\Delta\tau}} \left(\frac{1-\beta}{\beta}\right)^{\sigma\sqrt{M}\Delta\xi} \\ &\times e^{-M\Delta\tau(\beta\log\beta+(1-\beta)\log(1-\beta))} \beta^{d_2-d_1} (1-\beta)^{c_2-c_1+d_1-d_2} e^{-(\Delta\xi)^2/(2\Delta\tau)} (1 + O(1/\sqrt{M})) \end{aligned}$$

where we used (3.13).

Now consider the asymptotics of various parts of the integrand. We have

$$\frac{t_1!}{(t_2 - 1)!} = M(M\tau_1)^{M\tau_1+c_1} (M\tau_2)^{-M\tau_2-c_2} e^{-M\Delta\tau} \sqrt{\tau_1\tau_2} (1 + O(1/M)).$$

We can also write

$$\begin{aligned} \frac{(z\sigma\sqrt{M} - x_2 + 1)_{t_2-1}}{(w\sigma\sqrt{M} - x_1)_{t_1+1}} &= \frac{\Gamma(z\sigma\sqrt{M} - x_2 + t_2)}{\Gamma(z\sigma\sqrt{M} - x_2 + 1)} \frac{\Gamma(w\sigma\sqrt{M} - x_1)}{\Gamma(w\sigma\sqrt{M} - x_1 + t_1 + 1)} \\ &= (-1)^{x_1-x_2+1} \frac{\Gamma(z\sigma\sqrt{M} - x_2 + t_2)}{\Gamma(w\sigma\sqrt{M} - x_1 + t_1 + 1)} \frac{\Gamma(-z\sigma\sqrt{M} + x_2)}{\Gamma(1 - w\sigma\sqrt{M} + x_1)} \frac{\sin(\pi z\sigma\sqrt{M})}{\sin(\pi w\sigma\sqrt{M})}, \end{aligned}$$

where we used (3.12). This cancels with the existing ratio of the sine functions. Continuing with the asymptotics, we obtain

$$\begin{aligned} \frac{\Gamma(z\sigma\sqrt{M} - x_2 + t_2)}{\Gamma(w\sigma\sqrt{M} - x_1 + t_1 + 1)} \frac{\Gamma(-z\sigma\sqrt{M} + x_2)}{\Gamma(1 - w\sigma\sqrt{M} + x_1)} &= M^{-2} (M\tau_1)^{-M\tau_1-c_1} (M\tau_2)^{M\tau_2+c_2} e^{M\Delta\tau} \\ &\times e^{-M\Delta\tau(\beta\log\beta+(1-\beta)\log(1-\beta))} \beta^{d_2-d_1} (1-\beta)^{c_2-c_1+d_1-d_2} \left(\frac{1-\beta}{\beta}\right)^{\sigma\sqrt{M}(\Delta\xi+z-w)} \\ &\times \frac{1}{\beta(1-\beta)\tau_1\tau_2} \exp\left\{\frac{\tau_1(z-\xi_2)^2 - \tau_2(w-\xi_1)^2}{2\tau_1\tau_2}\right\} (1 + O(1/\sqrt{M})). \end{aligned}$$

We see that the factor

$$(-1)^{x_1-x_2} e^{-M\Delta\tau(\beta\log\beta+(1-\beta)\log(1-\beta))} \beta^{d_2-d_1} (1-\beta)^{c_2-c_1+d_1-d_2} \left(\frac{1-\beta}{\beta}\right)^{\sigma\sqrt{M}\Delta\xi}$$

appearing in both summands in (A.4) is of the form $f(\tau_1, \xi_1)/f(\tau_2, \xi_2)$, and thus does not affect the correlation functions (cf. footnote⁵). The remaining parts of (A.4) multiplied by $\sigma\sqrt{M}$ converge to (A.3), as desired. \square

Since the noncolliding Bernoulli random walks converge under the diffusion scaling to the Dyson Brownian Motion [EK08], the kernel (A.3) is the correlation kernel for the latter process started from the arbitrary initial configuration $(\alpha_1 \leq \dots \leq \alpha_N)$. When $\tau_1 = \tau_2$, the kernel (A.3) turns into the one appeared in [BH97], [Joh01b], [Shc09]. Utilizing (A.3), these papers show that the local statistics of the eigenvalues of the deformed GUE ensemble (equivalently, the distribution of the Dyson Brownian Motion started from $(\alpha_1 \leq \dots \leq \alpha_N)$ at a fixed time) are universally governed by the continuous sine kernel.

APPENDIX B. REPRESENTATION-THEORETIC INTERPRETATION OF NONCOLLIDING WALKS

The random matrix analogue of noncolliding Bernoulli or Poisson random walks is the GUE Dyson Brownian Motion. The distribution of the Dyson Brownian Motion started from an arbitrary initial configuration $(\alpha_1 \leq \dots \leq \alpha_N) \in \mathbb{R}^N$ after time $t > 0$ can also be interpreted as the eigenvalue distribution of the deformed GUE ensemble $A + \sqrt{t}G$, where $A = \text{diag}(\alpha_1, \dots, \alpha_N)$ is a fixed diagonal matrix, and G is an $N \times N$ random matrix from the GUE.¹⁶ Let us discuss a similar interpretation of the noncolliding Bernoulli or Poisson random walks in terms of representation theory of unitary groups.

Irreducible representations of the unitary group $U(N)$ can be parametrized by points of \mathbb{W}^N . Let $\chi_{\vec{x}}(u_1, \dots, u_N)$, $\vec{x} \in \mathbb{W}^N$ denote the corresponding normalized irreducible characters. Here the u_i 's are eigenvalues of the matrix from $U(N)$, and the characters $\chi_{\vec{x}}$ are normalized in the sense that $\chi_{\vec{x}}(1, \dots, 1) = 1$. These characters are the normalized Schur polynomials:

$$\chi_{\vec{x}}(u_1, \dots, u_N) = \frac{s_{\lambda}(u_1, \dots, u_N)}{\dim_N \vec{x}},$$

where $\lambda = (\lambda_1 \geq \dots \geq \lambda_N)$ with $\lambda_i = x_{N+1-i} + i - N$ is the highest weight of the representation, and

$$\dim_N \vec{x} = s_{\lambda}(1, \dots, 1) = \prod_{1 \leq i < j \leq N} \frac{x_j - x_i}{j - i} = \frac{V(\vec{x})}{V(0, 1, \dots, N-1)}$$

is the dimension of this representation. Details on representations of unitary groups can be found in, e.g., [Wey97].

An (abstract) normalized character χ of $U(N)$ is defined as a nonnegative definite continuous function on $U(N)$ which satisfies $\chi(e) = 1$ and $\chi(ab) = \chi(ba)$ for any $a, b \in U(N)$ (that is, we speak about characters which do not necessarily correspond to actual representations). The set of such characters is convex, and the normalized irreducible characters are its extreme points. Thus, any abstract character can be decomposed into irreducibles as $\chi(u_1, \dots, u_N) = \sum_{\vec{x} \in \mathbb{W}^N} c_{\vec{x}} \chi_{\vec{x}}(u_1, \dots, u_N)$, where the numbers $\{c_{\vec{x}}\}_{\vec{x} \in \mathbb{W}^N}$ are nonnegative and sum to one, hence they define a probability distribution on \mathbb{W}^N .

The product of two normalized characters $\chi^{(1)}$ and $\chi^{(2)}$ is also a normalized character. (If both $\chi^{(1)}$ and $\chi^{(2)}$ correspond to actual representations, then $\chi^{(1)}\chi^{(2)}$ is the normalized character of the tensor product of these representations.) The product $\chi^{(1)}\chi^{(2)}$ then can be decomposed into irreducibles, thus yielding a probability distribution on \mathbb{W}^N .

Fix $\vec{a} \in \mathbb{W}^N$ and take the irreducible normalized character $\chi_{\vec{a}}$ as $\chi^{(1)}$. Let $\chi^{(2)}$ be the restriction to $U(N)$ of a certain extreme character of the infinite-dimensional unitary group $U(\infty)$. We will consider two classes of such characters of $U(N)$ having the form

$$\chi_{\beta;t}(u_1, \dots, u_N) = \prod_{r=1}^N (1 - \beta + \beta u_r)^t \quad \text{or} \quad \chi_{\tau}(u_1, \dots, u_N) = \prod_{r=1}^N e^{\tau(u_r - 1)},$$

¹⁶There are different normalizations of the GUE random matrices and Dyson Brownian Motion in the literature (cf. [AGZ10, (2.2.2)], [TV14, Example 2]), and here we assume that they agree.

where $t \in \mathbb{Z}_{\geq 0}$ and $\tau > 0$. There is a number of papers discussing classification of extreme characters of $U(\infty)$, e.g., see [Edr53], [Voi76], [VK82], [OO98], and other references in [BO12] and [Pet14b].

Proposition B.1. *The probability weights $\{c_{\vec{x}}\}_{\vec{x} \in \mathbb{W}^N}$ arising from the decomposition*

$$\chi_{\vec{a}}(u_1, \dots, u_N) \chi_{\beta; t}(u_1, \dots, u_N) = \sum_{\vec{x} \in \mathbb{W}^N} c_{\vec{x}} \chi_{\vec{x}}(u_1, \dots, u_N)$$

describe the distribution of the noncolliding Bernoulli random walk with parameter β started from the initial configuration \vec{a} after t steps.

Similarly, the decomposition of $\chi_{\vec{a}}(u_1, \dots, u_N) \chi_{\tau}(u_1, \dots, u_N)$ into irreducibles corresponds to the distribution of the noncolliding Poisson random walk started from the configuration \vec{a} after time τ .

The case $\vec{a} = (0, 1, \dots, N-1)$ leads to the trivial representation: $\chi_{\vec{a}}(u_1, \dots, u_N) \equiv 1$. In this case the distribution of the noncolliding Bernoulli or Poisson random walks is related to the decomposition of extreme characters of $U(\infty)$ into irreducibles. Probabilistic properties of the corresponding measures were studied in, e.g., [BK08], [BF14]. These measures can be regarded as discrete analogues of the GUE eigenvalue distribution.

Proof of Proposition B.1. This fact is well known to specialists, but we include its proof for completeness.

It suffices to consider only the Bernoulli $t = 1$ case, because the general t case follows by induction, and the Poisson statement follows by a simple limit transition. The result would follow if we interpret the coefficients c_{μ} in the decomposition

$$\frac{s_{\lambda}(u_1, \dots, u_N)}{s_{\lambda}(1, \dots, 1)} \prod_{r=1}^N (\beta u_r + 1 - \beta) = \sum_{\mu_1 \geq \dots \geq \mu_N} c_{\mu} \frac{s_{\mu}(u_1, \dots, u_N)}{s_{\mu}(1, \dots, 1)}, \quad \lambda_i = a_{N+1-i} + i - N,$$

as one-step transition probabilities (1.2). Multiply the above decomposition by $s_{\lambda}(1, \dots, 1)$ and the Vandermonde in the u_i 's, and expand the determinants in

$$\det[u_i^{\lambda_j + N - j}]_{i,j=1}^N \prod_{r=1}^N (\beta u_r + 1 - \beta) = \sum_{\mu_1 \geq \dots \geq \mu_N} c_{\mu} \det[u_i^{\mu_j + N - j}]_{i,j=1}^N \frac{s_{\lambda}(1, \dots, 1)}{s_{\mu}(1, \dots, 1)}.$$

Because of the ordering in λ and μ , it suffices to consider the coefficient by $u_1^{\mu_1 + N - 1} \dots u_N^{\mu_N}$ in $u_1^{\lambda_1 + N - 1} u_2^{\lambda_2 + N - 2} \dots u_N^{\lambda_N}$ multiplied by $\prod_{r=1}^N (\beta u_r + 1 - \beta)$. Clearly, picking βu_r from each r th factor corresponds to the r th particle jumping by one to the right, while picking $(1 - \beta)$ means that this particle stays put. Particle collisions are not allowed because $\mu_1 + N - 1 > \dots > \mu_N$, and the factor $s_{\lambda}(1, \dots, 1)/s_{\mu}(1, \dots, 1)$ can be identified with the ratio of the Vandermondes in (1.2).

The Poisson case follows from the above argument in the limit as $\beta \searrow 0$. \square

We see that tensor multiplication of representations (and, more generally, multiplication of normalized characters) is a discrete analogue of the matrix addition. Moreover, multiplying by a suitable extreme character of $U(\infty)$ corresponds to adding a multiple of the

GUE matrix. This similarity can be continued further to include the operation of the free convolution — its discrete analogue is the so-called quantized free convolution, see [BG15].

Therefore, our main universality results in Section 2.3 can be reformulated as bulk universality for tensor products of two representations of $U(N)$, when one of the factors is arbitrary, and the other factor is the specific representation $\mathbf{x}_{\beta;t}$ (with t large). We *conjecture* that under mild technical conditions the same bulk universality should hold for tensor products of two arbitrary representations of $U(N)$ (cf. [BES15] for a progress towards a similar random matrix result). A weaker version of the bulk universality for tensor products of two arbitrary representations is established in [Gor16].

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